

Kac–Moody symmetric spaces

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Abstract

In the present article we introduce and study a class of topological reflection spaces that we call Kac–Moody symmetric spaces. These are associated with split real Kac–Moody groups and generalize Riemannian symmetric spaces of non-compact type.

Based on preliminary work by the third-named author we observe that in a non-spherical Kac–Moody symmetric space there exist pairs of points that do not lie on a common geodesic; however, any two points can be connected by a chain of geodesic segments. We moreover classify maximal flats in Kac–Moody symmetric spaces and study their intersection patterns, leading to a classification of global and local automorphisms. Some of our methods apply to general topological reflection spaces beyond the Kac–Moody setting.

Unlike Riemannian symmetric spaces, non-spherical non-affine irreducible Kac–Moody symmetric spaces also admit an invariant causal structure. For causal and anti-causal geodesic rays with respect to this structure we find a notion of asymptoticity, which allows us to define a future and past boundary of such Kac–Moody symmetric space. We show that these boundaries carry a natural simplicial structure and are simplicially isomorphic to the halves of the geometric realization of the twin buildings of the underlying split real Kac–Moody group. We also show that every automorphism of the symmetric space is uniquely determined by the induced simplicial automorphism of the future and past boundary.

The invariant causal structure on a non-spherical non-affine irreducible Kac–Moody symmetric space gives rise to an invariant pre-order on the underlying space, and thus to a sub-semigroup of the Kac–Moody group. For many Kac–Moody symmetric spaces including the E_n -series, $n \geq 10$, we establish that this pre-order is actually a partial order. The case of general Kac–Moody symmetric spaces remains open.

We conclude that while in some aspects Kac–Moody symmetric spaces closely resemble Riemannian symmetric spaces, in other aspects they behave similarly to ordered affine hovels, their non-Archimedean cousins.

1 Introduction

Kac–Moody groups over a local field \mathbb{K} as for instance studied in [Rou06], [GR08], [HKM13], [GR14], [HK15], [BGR16] are infinite-dimensional generalizations of the groups of \mathbb{K} -points of (split) semisimple algebraic groups. From a geometric point of view, semisimple groups over local fields arise as subgroups of the isometry groups of Riemannian symmetric spaces (in the Archimedean case) and Euclidean buildings (in the non-Archimedean case). It is thus natural to ask whether Kac–Moody groups over local fields admit a similar geometric interpretation.

For Kac–Moody groups over non-Archimedean local fields such a geometric interpretation is described in [Rou11], where the author discusses the notion of a *measure affine ordonnée* (sometimes translated as *ordered affine hovel* into English, e.g. in [GR08]). *Measures* are certain generalizations of Euclidean buildings that admit an action by a Kac–Moody group over a non-Archimedean local field \mathbb{K} , generalizing the notion of a Bruhat–Tits building endowed with the action of the \mathbb{K} -points of a split semisimple group.

In the present article we investigate the Archimedean situation, focussing on the split real case. We introduce a generalization of Riemannian symmetric spaces, which we call Kac–Moody symmetric spaces and on which split real Kac–Moody groups act in a way that generalizes the action of semisimple split real Lie groups on their Riemannian symmetric spaces. It turns out that in this setting one can observe both phenomena that one is familiar with from the finite-dimensional theory and phenomena that are specific to the infinite-dimensional situation; some of these infinite-dimensional phenomena in fact have non-Archimedean analogs in the theory of *masures*.

A key structural problem that one has to face when generalizing the notion of a Riemannian symmetric space, is that the latter is originally defined in terms of a smooth Riemannian metric on a manifold; we are unaware of any reasonable notion of smoothness on the kind of homogeneous spaces on which a (non-spherical and non-affine) real Kac–Moody group naturally acts, nor are these spaces metrizable with respect to their natural topologies. Our starting point is thus an alternative characterization of affine symmetric spaces, due to Ottmar Loos [Loo67a, Loo67b].

Fact 1.1 (Loos [Loo67a, Loo67b]). *Let \mathcal{X} be an affine symmetric space, and given $x, y \in \mathcal{X}$ denote by $x \cdot y$ the point reflection of y at x . Then $\mu : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$, $\mu(x, y) := x \cdot y$ is a C^1 -map satisfying the following axioms:*

- (RS1) *for any $x \in \mathcal{X}$ we have $x \cdot x = x$,*
- (RS2) *for any pair of points $x, y \in \mathcal{X}$ we have $x \cdot (x \cdot y) = y$,*
- (RS3) *for any triple of points $x, y, z \in \mathcal{X}$ we have $x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z)$,*
- (RS4_{loc}) *every $x \in \mathcal{X}$ has a neighbourhood U such that $x \cdot y = y$ implies $y = x$ for all $y \in U$.*

Conversely, if \mathcal{X} is a smooth manifold and $\mu : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$ is a C^1 -map subject to (RS1)–(RS4_{loc}) above, then \mathcal{X} is an affine symmetric space, and $\mu(x, y)$ is the point reflection of y at x . If \mathcal{X} is a Riemannian symmetric space, then the isometries of \mathcal{X} are exactly the C^1 -maps $\alpha : \mathcal{X} \rightarrow \mathcal{X}$ satisfying $\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y)$. If \mathcal{X} is moreover of the non-compact type, then instead of the local condition (RS4_{loc}) it satisfies the global condition

- (RS4) *$x \cdot y = y$ implies $y = x$ for all $y \in U$.*

Since we are interested in generalizations of Riemannian symmetric spaces of non-compact type, we define the following:

Definition 1.2. A pair (\mathcal{X}, μ) is called a *topological symmetric space* provided \mathcal{X} is a topological space and $\mu : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$, $\mu(x, y) := x \cdot y$ is a continuous map subject to the axioms (RS1)–(RS4) above. The *automorphism group* $\text{Aut}(\mathcal{X}, \mu)$ of (\mathcal{X}, μ) is defined as

$$\text{Aut}(\mathcal{X}, \mu) := \{\alpha : \mathcal{X} \rightarrow \mathcal{X} \mid \alpha \text{ homeomorphism, } \alpha(x \cdot y) = \alpha(x) \cdot \alpha(y)\}.$$

Loos’ theorem strongly uses the differentiability of μ , and not much is known about general topological symmetric spaces without any smoothness assumption. For example, it is not even known to us whether a topological symmetric space which is homeomorphic to a finite-dimensional manifold necessarily arises from an affine symmetric space.

We thus pursue three goals in the present article:

- (i) to develop a basic theory of topological symmetric spaces in the absence of any smoothness assumption;
- (ii) to associate a topological symmetric space to a large class of Kac–Moody groups over an Archimedean local field (focusing on the split real case for simplicity);

- (iii) to develop the structure theory of such Kac–Moody symmetric spaces, studying their geodesics, maximal flats, (local and global) automorphisms, causal structures and boundaries.

Our results concerning (i) might actually be of interest beyond Kac–Moody theory.

The three concepts of flats, geodesics and one-parameter subgroups of the isometry group are of fundamental nature in the study of Riemannian symmetric spaces. The former two are usually defined using the curvature tensor, and the existence of the latter is derived from an existence theorem for ordinary differential equations. In our topological setting we need to define flats and geodesics without reference to the curvature tensor, and to establish the existence of one-parameter subgroups without analytic tools.

Given a topological symmetric space (\mathcal{X}, μ) we call a subset $\gamma \subset \mathcal{X}$ a *geodesic* if there exists a bijection $\varphi : \mathbb{R} \rightarrow \gamma$ such that $\varphi(2x - y) = \mu(\varphi(x), \varphi(y))$ for all $x, y \in \mathbb{R}$. Compact connected subsets of geodesics will be called *geodesic segments*. By the following result, geodesics give rise to one-parameter subgroups of $\text{Aut}(\mathcal{X}, \mu)$.

Theorem 1.3 (Existence of one-parameter subgroups without differentiability assumptions; cf. Proposition 2.19). *Let (\mathcal{X}, μ) be a topological symmetric space. Given $x \in \mathcal{X}$ let $s_x(y) := \mu(x, y)$ and given a geodesic $\gamma \subset \mathcal{X}$ let*

$$T_\gamma := \{s_p \circ s_q \mid p, q \in \gamma\} \subset \text{Aut}(\mathcal{X}, \mu).$$

- (i) $T_\gamma \cong (\mathbb{R}, +)$ is a one-parameter subgroup of $\text{Aut}(\mathcal{X}, \mu)$.
- (ii) T_γ acts sharply transitively on γ by Euclidean translations.
- (iii) If $t_1, t_2 \in T_\gamma$ and $t_1|_\gamma = t_2|_\gamma$, then $t_1 = t_2$.
- (iv) If any two points in \mathcal{X} can be connected by a finite chain of geodesic segments, then the one-parameter subgroups T_γ generate a subgroup of $\text{Aut}(\mathcal{X}, \mu)$ of index ≤ 2 .

As for the definition of a flat in a topological symmetric space, we offer two notions, which we will later show to lead to the same concept in Kac–Moody symmetric spaces. Firstly, we have the following purely synthetic definition:

Definition 1.4. A closed subset $F \subset \mathcal{X}$ of cardinality ≥ 2 is called a *weak flat* if it satisfies the following properties:

- (F1) F is a *reflection subspace*, i.e. if $x, y \in F$, then $x \cdot y \in F$.
- (F2) F is *midpoint convex*, i.e. if $x, y \in F$ then there exists $z \in F$ with $z \cdot x = y$ (and thus $z \cdot y = x$).
- (F3) F is *weakly abelian*, i.e. for all $x, y, z \in F$ one has

$$x \cdot (z \cdot (y \cdot z)) = y \cdot (z \cdot (x \cdot z)).$$

Alternatively, denote by $\mathbb{E}^n = (\mathbb{R}^n, \mu)$ the Euclidean symmetric space given by $\mu(x, y) := 2x - y$. We call a closed reflection subspace F of a topological symmetric space \mathcal{X} a *Euclidean flat* of *rank* n if it is isomorphic to \mathbb{E}^n as a topological reflection space. Then geodesics are just Euclidean flats of rank 1, and every Euclidean flat is a weak flat, see Figure 1.

We now turn to the main objects of our interest in the present article and introduce Kac–Moody symmetric spaces, a class of topological symmetric spaces associated with (real split) Kac–Moody groups. Given a generalized Cartan matrix \mathbf{A} (see Definition 3.2) we denote by $G = G(\mathbf{A})$ the corresponding *simply connected centered split real Kac–Moody group* of type \mathbf{A} (see Definition 3.4). Throughout this article we will assume that \mathbf{A} is *irreducible* and *symmetrizable* (see Definition 3.2), and we will consider G as a topological group with the *Kac–Petersen topology*

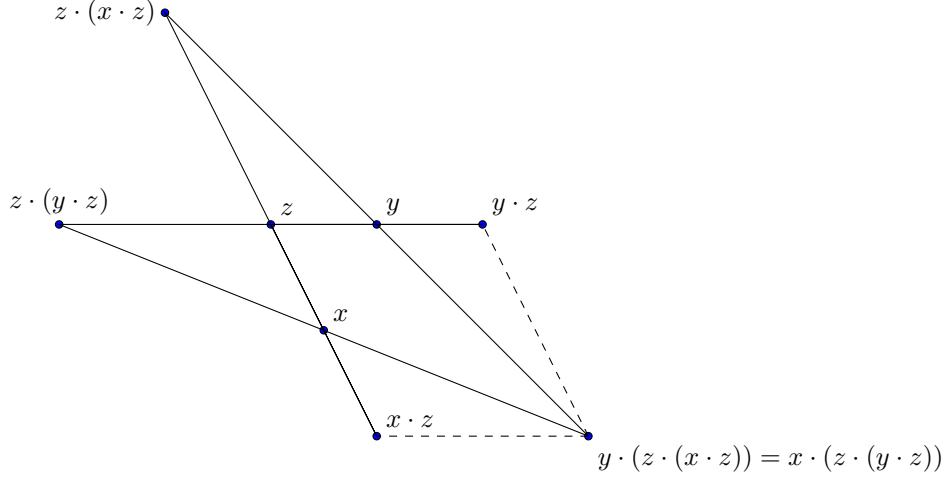


Figure 1: Euclidean space is weakly abelian.

(see Definition 3.4). For some of our results we will need additional assumptions on \mathbf{A} (e.g. non-spherical, non-affine, on a few occasions two-spherical), but for the basic definitions we do not need any of these assumptions.

There exists a canonical continuous involution θ of G which on each standard rank one subgroup restricts to the involution $g \mapsto g^{-\top}$. Any involution conjugate to this involution in the semidirect product $G \rtimes \langle \theta \rangle$ is called a *Cartan–Chevalley involution*. The group G acts transitively by conjugation on the set \mathcal{X}_G of Cartan–Chevalley involutions, and we equip \mathcal{X}_G with the quotient topology with respect to this action.

Proposition 1.5 (Cf. Theorem 4.12). *The space \mathcal{X}_G is a topological symmetric space with respect to*

$$\mu : \mathcal{X}_G \times \mathcal{X}_G \rightarrow \mathcal{X}_G, \quad \mu(\alpha, \beta) := \alpha \circ \beta \circ \alpha.$$

Definition 1.6. The symmetric space (\mathcal{X}_G, μ) is called the *unreduced Kac–Moody symmetric space* of the real split Kac–Moody group G .

In the spherical case, i.e., if the Kac–Moody group G actually is a Lie group, this is the (involution model of the) associated Riemannian symmetric space.

If the Cartan matrix \mathbf{A} is non-invertible, then the center $Z(G)$ of G has positive dimension, given by the corank of \mathbf{A} . In this case, the unreduced Kac–Moody symmetric space \mathcal{X}_G fibers over a topological symmetric space $\overline{\mathcal{X}}_G$ with fiber given by a Euclidean space of dimension equal to the corank of \mathbf{A} , and the adjoint quotient $\text{Ad}(G)$ of G acts on $\overline{\mathcal{X}}_G$. We refer to $\overline{\mathcal{X}}_G$ as the *reduced Kac–Moody symmetric space* of G . In the case where \mathbf{A} is non-invertible, it is this reduced version that resembles most closely a Riemannian symmetric space.

The following results describes flats in Kac–Moody symmetric spaces.

Theorem 1.7 (Flats in Kac–Moody symmetric spaces; cf. Section 5C). *Let \mathcal{X}_G be an unreduced Kac–Moody symmetric space, and let $\overline{\mathcal{X}}_G$ be its reduced quotient.*

- (i) *Every weak flat in \mathcal{X}_G or $\overline{\mathcal{X}}_G$ is Euclidean. In particular, all weak flats are finite-dimensional and locally compact.*
- (ii) *Every weak flat in \mathcal{X}_G or $\overline{\mathcal{X}}_G$ is contained in a maximal weak flat.*
- (iii) *The projection $\mathcal{X}_G \rightarrow \overline{\mathcal{X}}_G$ induces a bijection between maximal weak flats in \mathcal{X}_G and maximal weak flats in $\overline{\mathcal{X}}_G$.*

(iv) G acts transitively on pairs (p, F) where F is a maximal weak flat in \mathcal{X}_G (or $\overline{\mathcal{X}}_G$) and $p \in F$. In particular, all maximal weak flats in \mathcal{X}_G (respectively $\overline{\mathcal{X}}_G$) are Euclidean spaces of the same dimension $r(\mathcal{X}_G)$ (respectively $r(\overline{\mathcal{X}}_G)$).

(v) $r(\mathcal{X}_G)$ equals the size of \mathbf{A} , and $r(\overline{\mathcal{X}}_G)$ equals the rank of \mathbf{A} .

The integers $r(\mathcal{X}_G)$ and $r(\overline{\mathcal{X}}_G)$ are called the *rank* of \mathcal{X}_G and $\overline{\mathcal{X}}_G$ respectively. In the sequel we refer to a maximal weak flat simply as a *maximal flat*, and to a pair (p, F) as in (iii) as a *pointed maximal flat*. Besides maximal flats, we are also interested in minimal non-trivial flats, i.e. geodesics.

Theorem 1.8 (Geodesic connectedness of Kac–Moody symmetric spaces; cf. Section 5B). *The Kac–Moody symmetric spaces \mathcal{X}_G and $\overline{\mathcal{X}}_G$ have the following properties:*

- (i) \mathcal{X}_G and $\overline{\mathcal{X}}_G$ are *geodesically connected*, i.e. any two points in \mathcal{X}_G or $\overline{\mathcal{X}}_G$ can be connected by a finite chain of geodesic segments.
- (ii) If \mathbf{A} is not spherical, then \mathcal{X}_G and $\overline{\mathcal{X}}_G$ are *non-geodesic*, i.e. there exist points $x, y \in \mathcal{X}_G$ (and also in $\overline{\mathcal{X}}_G$) which do not lie on a common geodesic (and hence are not contained in a common maximal flat).

Note that (ii) is in stark contrast to the case of Riemannian symmetric spaces, which are always geodesic. It is, however, reminiscent of the corresponding property of *maasures*: not every pair of points is contained in a common apartment. In fact, this property is the key feature that separates the class of *maasures* from the class of buildings.

By construction, the group G acts by automorphisms on \mathcal{X}_G and thus on its quotient $\overline{\mathcal{X}}_G$. The latter action (but in general not the former) factors through a faithful action of $\text{Ad}(G)$. As in the spherical case, the full automorphism group of $\overline{\mathcal{X}}_G$ is slightly larger than $\text{Ad}(G)$.

Theorem 1.9 (Automorphisms of reduced Kac–Moody symmetric spaces; cf. Corollary 6.3 and Theorem 6.5). *The group $\text{Ad}(G)$ is a finite index subgroup of the automorphism group $\text{Aut}(\overline{\mathcal{X}}_G)$. More precisely, $\text{Aut}(\overline{\mathcal{X}}_G)$ is isomorphic to $\text{Aut}(G) \cong \text{Aut}(\text{Ad}(G))$, and every automorphism in $\text{Aut}(\text{Ad}(G))$ can be written as a product of an inner automorphism, a diagonal automorphism, a power of a fixed Cartan–Chevalley involution and an automorphism of the Dynkin diagram $\Gamma_{\mathbf{A}}$. Moreover, $\text{Aut}(\overline{\mathcal{X}}_G)$ embeds into the automorphism group of the twin building associated with G and if \mathbf{A} is non-spherical then*

$$\text{Aut}(\overline{\mathcal{X}}_G) = \text{Aut}^+(\overline{\mathcal{X}}_G) \rtimes \langle s_o \rangle,$$

where $\text{Aut}^+(\overline{\mathcal{X}}_G) < \text{Aut}(\overline{\mathcal{X}}_G)$ is the index two subgroup preserving the two halves of the twin building (instead of interchanging the two halves).

Convention 1.10. *For the rest of this introduction we assume that \mathbf{A} is non-spherical and non-affine (on top of our standing assumptions that \mathbf{A} be irreducible and symmetrizable).*

Besides the global automorphisms in $\text{Aut}(\overline{\mathcal{X}}_G)$ one can also consider local automorphisms of $\overline{\mathcal{X}}_G$ in the following sense. If (p, F) is a pointed maximal flat in $\overline{\mathcal{X}}_G$, then we denote by $F^{\text{sing}}(p) \subset F$ the subset of points of F which are contained in more than one maximal flat containing p . We also fix an isomorphism $\varphi : \mathbb{E}^r \rightarrow F$ of reflection spaces with $\varphi(0) = p$. Then by Proposition 2.17 the *local automorphism group*

$$\text{Aut}(p, F) := \{ \alpha : F \rightarrow F \mid \alpha(F^{\text{sing}}(p)) = F^{\text{sing}}(p) \text{ and } \hat{\alpha} := \varphi^{-1} \circ \alpha \circ \varphi \in \text{GL}_n(\mathbb{R}) \}$$

does not depend on φ and up to isomorphism is independent of the choice of pointed maximal flat (p, F) . The relation between global and local automorphisms is as follows. Let

$$\begin{aligned} \text{Stab}(p, F) &:= \{ g \in \text{Aut}(\overline{\mathcal{X}}_G) \mid g.F = F, g.p = p \}, \\ \text{Fix}(p, F) &:= \{ g \in \text{Aut}(\overline{\mathcal{X}}_G) \mid \forall f \in F : g.f = f \}, \\ W(\text{Aut}(\overline{\mathcal{X}}_G) \curvearrowright \overline{\mathcal{X}}_G) &:= \text{Stab}(p, F) / \text{Fix}(p, F), \end{aligned}$$

and define $\text{Stab}_G(p, F)$, $\text{Fix}_G(p, F)$ and $W(G \curvearrowright \overline{\mathcal{X}}_G)$ similarly by restricting to automorphisms in G . Then $W(\text{Aut}(\overline{\mathcal{X}}_G) \curvearrowright \overline{\mathcal{X}}_G)$ and $W(G \curvearrowright \overline{\mathcal{X}}_G)$ are subgroups of the local automorphism group acting faithfully on (p, F) . Our next goal is to describe these actions more explicitly.

Recall that one can associate the generalized Cartan matrix \mathbf{A} with a Coxeter system (W, S) whose Coxeter diagram $\Gamma_{(W, S)}$ has the same underlying graph as the Dynkin diagram $\Gamma_{\mathbf{A}}$ of \mathbf{A} , but whose labelling carries less information (see Subsection AC). Also recall from Lemma A.16 that the automorphism group of the Coxeter complex $\Sigma = \Sigma(W, S)$ is given by

$$\text{Aut}(\Sigma(W, S)) = W \rtimes \text{Aut}(\Gamma_{(W, S)}).$$

Theorem 1.11 (Local vs. global automorphisms; cf. Theorem 6.10). *Let $\overline{\mathcal{X}}_G$ be a reduced Kac–Moody symmetric space.*

- (i) *The local automorphism group $\text{Aut}(p, F)$ of $\overline{\mathcal{X}}_G$ is isomorphic to $\text{Aut}(\Sigma(W, S)) \times \mathbb{Z}/2\mathbb{Z}$.*
- (ii) *Under this isomorphism the subgroup $W(\text{Aut}(\overline{\mathcal{X}}_G) \curvearrowright \overline{\mathcal{X}}_G) < \text{Aut}(p, F)$ corresponds to the finite index subgroup $(W \rtimes \text{Aut}(\Gamma_{\mathbf{A}})) \times \mathbb{Z}/2\mathbb{Z} < (W \rtimes \text{Aut}(\Gamma_{(W, S)})) \times \mathbb{Z}/2\mathbb{Z}$.*
- (iii) *Every local automorphism is the restriction of a global automorphism if and only if every automorphism of $\Gamma_{(W, S)}$ induces an isomorphism of $\Gamma_{\mathbf{A}}$. For instance, this happens if $\Gamma_{\mathbf{A}}$ is simply-laced.*

Corollary 1.12 (Algebraic Weyl group equals geometric Weyl group; cf. Theorem 6.11). *The geometric Weyl group $W(G \curvearrowright \overline{\mathcal{X}}_G)$ of G is isomorphic to the algebraic Weyl group W .*

With the notable exception of Theorem 1.8.(ii) the part of the theory of Kac–Moody symmetric spaces described so far follows closely the classical theory of Riemannian symmetric spaces. On the other hand, it turns out that (irreducible, non-spherical, non-affine) Kac–Moody symmetric spaces also carry additional structure, which is not shared by Riemannian symmetric spaces of the non-compact type, but which is shared by a different class of affine symmetric spaces called *causal symmetric spaces* (see [HÓ97]).

Proposition 1.13 (Existence of an invariant causal structure, cf. Section 7A). *There exists a canonical family $(\overline{C}_x^+)_{x \in \overline{\mathcal{X}}_G}$ of subsets of $\overline{\mathcal{X}}_G$ with the following properties:*

- (i) *$(\overline{C}_x^+)_{x \in \overline{\mathcal{X}}_G}$ is a *cone field*, i.e. for every $x \in \overline{\mathcal{X}}_G$ the subset $\overline{C}_x^+ \subset \overline{\mathcal{X}}_G$ intersects every flat containing x in an open cone with tip x .*
- (ii) *$(\overline{C}_x^+)_{x \in \overline{\mathcal{X}}_G}$ is *invariant* under $\text{Aut}^+(\overline{\mathcal{X}}_G)$, i.e. $\alpha(\overline{C}_x^+) = \overline{C}_{\alpha(x)}^+$ for all $\alpha \in \text{Aut}^+(\overline{\mathcal{X}}_G)$ and $x \in \overline{\mathcal{X}}_G$.*

In analogy with the theory of causal symmetric spaces we refer to the invariant cone field $(\overline{C}_x^+)_{x \in \overline{\mathcal{X}}_G}$ as the *canonical causal structure* on $\overline{\mathcal{X}}_G$. Roughly speaking, the canonical causal structure is a global version of the Tits cone in the underlying Kac–Moody Lie algebra. For the precise definition see Subsection 7A.

From the canonical causal structure we infer a notion of causal (or “time-like¹”) curve in $\overline{\mathcal{X}}_G$. Namely, we say that a continuous curve $\gamma : [S, T] \rightarrow \overline{\mathcal{X}}$ with $0 < S < T < \infty$ is *causal* if for every $t \in [S, T]$ there exists $\varepsilon > 0$ such that

$$\gamma((t, t + \varepsilon)) \subset \overline{C}_{\gamma(t)}^+.$$

The notion of an *anti-causal curve* can be defined dually. Using causal geodesic rays in $\overline{\mathcal{X}}_G$ we associate two further structures with $\overline{\mathcal{X}}_G$ which have no counterpart in the theory of Riemannian

¹In the study of Lorentzian causal structures, causal curves are also called *time-like curves*. Since the causal structures investigated here need not be Lorentzian, we will not use this terminology.

symmetric spaces, but which are reminiscent to classical objects in the theory of causal symmetric spaces: The *causal boundary* of $\overline{\mathcal{X}}_G$ and the *causal pre-order* on $\overline{\mathcal{X}}_G$.

Let us first describe the construction and basic properties of the causal boundary. Denote by $\partial_\bullet \overline{\mathcal{X}}_G$ the collection of geodesic rays in $\overline{\mathcal{X}}_G$, and by $\Delta_\bullet^\pm \subset \partial_\bullet \overline{\mathcal{X}}_G$ the subset of all causal/anti-causal geodesic rays. By invariance of the causal structure, the subsets Δ_\bullet^\pm are invariant under $\text{Aut}^+(\overline{\mathcal{X}})$ and their union Δ_\bullet is invariant under $\text{Aut}(\overline{\mathcal{X}})$. Points in the causal boundary will be defined as equivalence classes of causal or anti-causal rays by an equivalence relation which mimics asymptoticity of geodesic rays in Riemannian symmetric spaces.

Recall that if \mathcal{X} is a non-compact Riemannian symmetric space, then two geometric rays in \mathcal{X} are called *asymptotic*, if they are at bounded Hausdorff distance. For example, two geodesic rays r_1, r_2 in Euclidean space \mathbb{E}^n are asymptotic if and only if they are parallel and point in the same direction, i.e. they are of the form $r_1(t) = x + tv$ and $r_2(t) = y + tv$ for some $x, y \in \mathbb{R}^n$ and a unit vector v , and two geodesic rays in the hyperbolic plane are asymptotic if they have the same endpoint in the boundary. In Subsection 7E we construct equivalence relations \parallel on Δ_\bullet^\pm with the following properties:

- (A1) If $r \in \Delta_\bullet^\pm$ and $x \in \overline{\mathcal{X}}_G$, then there exists a unique $r' \in \Delta_\bullet^\pm$ emanating from x with $r \parallel r'$.
- (A2) \parallel is invariant under $\text{Aut}^+(\overline{\mathcal{X}})$, i.e. if $r_1 \parallel r_2$, then $\alpha(r_1) \parallel \alpha(r_2)$ for all $\alpha \in \text{Aut}^+(\overline{\mathcal{X}})$.
- (A3) If $r_1, r_2 \in \Delta_\bullet^\pm$ are contained in a common embedded hyperbolic plane in $\overline{\mathcal{X}}_G$, which arises as the orbit of a rank one subgroup of G , then $r_1 \parallel r_2$ if and only if they are asymptotic in the hyperbolic sense.
- (A4) If $r_1, r_2 \in \Delta_\bullet^\pm$ are contained in a common maximal flat F , then $r_1 \parallel r_2$ if and only if they are asymptotic in the Euclidean sense.

In view of these properties we call $r_1, r_2 \in \Delta_\bullet^\pm$ *asymptotic* if $r_1 \parallel r_2$.

Definition 1.14. The set $\Delta_\parallel^+ := \Delta_\bullet^+ / \parallel$ of asymptoticity classes of causal rays is called the *future boundary* of the Kac–Moody symmetric space $\overline{\mathcal{X}}_G$, and the set $\Delta_\parallel^- := \Delta_\bullet^- / \parallel$ is called its *past boundary*. The union $\Delta_\parallel := \Delta_\parallel^+ \sqcup \Delta_\parallel^-$ is called the *causal boundary*.

By (A2) the $\text{Aut}^+(\overline{\mathcal{X}}_G)$ -action on causal/anti-causal curves induces an action on the future/past boundary. In Subsection 7C we construct an $\text{Aut}^+(\overline{\mathcal{X}}_G)$ -invariant simplicial structure on each of these boundaries, using only the geometry of flats in $\overline{\mathcal{X}}_G$.

Theorem 1.15 (Twin building vs. causal boundary; cf. Corollary 7.27). *The causal boundary Δ_\parallel is simplicially and $\text{Aut}^+(\overline{\mathcal{X}}_G)$ -equivariantly isomorphic to the geometric realization $|\Delta|$ of the twin building Δ of G , and thus the past and future boundary are simplicially and $\text{Aut}^+(\overline{\mathcal{X}}_G)$ -equivariantly isomorphic to the geometric realizations of the halves of this twin building.*

Theorem 1.15 should be compared to the classical fact that the geometric boundary of a Riemannian symmetric space of non-compact type, i.e. the collection of all geodesic rays modulo asymptoticity, carries a natural simplicial structure which is isomorphic to the geometric realization of the corresponding spherical building (see, e.g., [KL97]). This analogy is meaningful, since in the finite-dimensional case, the Tits cone is given by the whole Cartan subalgebra, and hence the canonical causal structure is the trivial causal structure in which every curve is causal.

In the case of a hyperbolic Kac–Moody group, Theorem 1.15 can be seen as a global version of the lightcone embedding of the twin building as described in [CFF16]. The analogous construction of a twin building at infinity for *measures* can be found in [Rou11, Section 3]; by [CMR17, Theorem 1] this twin building at infinity of a *measure* actually carries a natural topology that turns it into a weak topological twin building in the sense of [HKM13].

As in the finite-dimensional case, each asymptoticity class of causal rays in a Kac–Moody symmetric space forms an orbit under the action of an appropriate parabolic subgroup of G (see

Proposition 7.24). Geometrically this means that if r is a causal ray, which is regular in the sense that it is contained in a unique maximal flat, then all the causal rays parallel to r can be obtained by parallel-translating r in this flat and then sliding the resulting rays along suitable horospheres.

To push the analogy with the Riemannian case even further, recall that every automorphism of a Riemannian symmetric space is uniquely determined by its action on the geometric boundary, i.e., the spherical building at infinity. In the Kac–Moody setting a similar statement is true: The automorphism is uniquely determined by its action on the causal boundary, i.e., the twin building at infinity.

Theorem 1.16 (Causal boundary rigidity; cf. Corollary 7.27). *Every automorphism of $\overline{\mathcal{X}}_G$ is uniquely determined by the induced simplicial automorphism of the causal boundary. Every automorphism in $\text{Aut}^+(\overline{\mathcal{X}})$ is uniquely determined by the induced simplicial automorphism of the future (or past) boundary.*

Having discussed the causal boundary of Kac–Moody symmetric spaces, we now turn to the second structure on $\overline{\mathcal{X}}_G$ induced by the canonical causal structure: We write $x \prec y$ and say that x *strictly causally precedes* y if there exists a piecewise geodesic causal curve $\gamma : [S, T] \rightarrow \overline{\mathcal{X}}$ with $\gamma(S) = x$ and $\gamma(T) = y$, and we define the *causal pre-order* \preceq on $\overline{\mathcal{X}}_G$ by setting $x \preceq y$ if $x \prec y$ or $x = y$.

Invariance of the causal structure implies that the pre-order \preceq is invariant under $\text{Aut}^+(\overline{\mathcal{X}}_G)$. It is currently not known whether the causal pre-order of an irreducible, non-spherical, non-affine Kac–Moody symmetric space is always anti-symmetric, i.e. a partial order. However, we are able to establish this property in many examples of interest, including the E_n -series for $n \geq 10$, the AE_n -series for $n \geq 5$, and Kac–Moody groups of all but three non-compact hyperbolic types. The unifying feature of all these examples is their star-sphericity. Here an irreducible non-spherical non-affine Kac–Moody group G is called *star-spherical* if for every vertex v in the underlying Dynkin diagram $\Gamma_{\mathbf{A}}$ the star $\text{st}(v)$ of v in $\Gamma_{\mathbf{A}}$ is a Dynkin diagram of spherical type.

Theorem 1.17 (Causal order for star-spherical Kac–Moody symmetric spaces; cf. Theorem 8.8). *If G is star-spherical, then the causal pre-order \preceq on $\overline{\mathcal{X}}_G$ is a partial order.*

In general, anti-symmetry of \preceq depends on a certain convexity property of G . The assumption that G is star-spherical ensures that for every vertex $v \in \Gamma_{\mathbf{A}}$ the subgroup G_v of G generated by the rank one subgroups corresponding to the vertices of $\text{st}(v)$ is a quasi-simple Lie group. One then establishes the necessary convexity properties of G by applying Kostant’s classical convexity theorem [Kos73, Theorem 4.1] to the subgroups G_v . It is currently an open problem whether Kostant’s convexity theorem holds for general Kac–Moody groups. An infinitesimal version was established by Kac and Peterson in [KP84] and the problem of extending Theorem 1.17 to arbitrary irreducible non-spherical non-affine Kac–Moody symmetric spaces is closely related to the question whether this infinitesimal version can be extended to a global convexity theorem on the group level in general.

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Contents

1	Introduction	1
2	Concepts from synthetic geometry	10
2A	Reflection spaces	10
2B	Involution model and quadratic representation	12
2C	Topological reflection spaces	13
2D	Flats in topological reflection spaces	14
2E	Geodesics and translation groups	15
2F	Geodesically connected reflection spaces	17
2G	Local automorphisms of strongly transitive reflection spaces	18
3	Split real Kac–Moody groups and their Iwasawa decompositions	19
3A	Groups with RGD systems	19
3B	Complex and split real topological Kac–Moody groups	20
3C	The adjoint quotient and the semisimple adjoint quotient	21
3D	The extended Weyl group	24
3E	The twin BN pair and the twin building	24
3F	The Cartan–Chevalley involution and the twist map	26
3G	The topological Iwasawa decomposition	29
3H	The image of the twist map	31
4	Models for Kac–Moody symmetric spaces	32
4A	Topological symmetric spaces from involutions	32
4B	Reduced and unreduced Kac–Moody symmetric spaces	33
4C	Reflections, transvections and reflection-homogeneity	35
4D	Models for Kac–Moody symmetric spaces	36
4E	Comparison of topologies	37
5	Flats and geodesics in Kac–Moody symmetric spaces	38
5A	Standard flats	38
5B	Midpoint convex subsets and geodesic connectedness	41
5C	The classification of maximal flats	42
6	Local and global automorphisms of Kac–Moody symmetric spaces	45
6A	Automorphisms of Kac–Moody groups	45
6B	Automorphisms of the main group	47
6C	Global automorphisms of reduced Kac–Moody symmetric spaces	47
6D	Local automorphisms and the Coxeter complex	48
6E	Local vs. global automorphisms	50

7	Causal structures and the causal boundary	51
7A	Invariant causal structures	51
7B	Causal geodesic rays and the municipality	53
7C	The simplicial structure of the municipality	55
7D	The global structure of the municipality	57
7E	Asymptoticity of causal and anti-causal rays	58
7F	The causal boundary	61
8	Geometry of the causal pre-order	62
8A	Causal curves and the causal pre-order	62
8B	Convexity conditions	63
8C	Causal partial orders on star-spherical Kac–Moody symmetric spaces	64
A	Complex Kac–Moody algebras and the Weyl group	66
AA	Complex Kac–Moody algebras	66
AB	The Kac–Moody representation of the Weyl group	68
AC	The Coxeter system of the Weyl group	69
AD	Root bases and Coxeter systems	70
AE	Root bases for Weyl groups with symmetrizable generalized Cartan matrix	70
AF	The reduced Tits cone and the Coxeter complex	72
AG	Automorphisms of the Coxeter complex acting on the Tits cone	73

2 Concepts from synthetic geometry

2A Reflection spaces

Definition 2.1. Let \mathcal{X} be a set and $\mu : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$, $(x, y) \mapsto x \cdot y$ be a map.

(i) (\mathcal{X}, μ) is called a *reflection space* if it satisfies the following axioms:

(RS1) for any $x \in \mathcal{X}$ one has $x \cdot x = x$,

(RS2) for any pair of points $x, y \in \mathcal{X}$ one has $x \cdot (x \cdot y) = y$,

(RS3) for any triple of points $x, y, z \in \mathcal{X}$ one has $x \cdot (y \cdot z) = (x \cdot y) \cdot (x \cdot z)$.

(ii) A reflection space is called *symmetric* or a *symmetric space* if it satisfies the additional axiom:

(RS4) $x \cdot y = y$ implies $y = x$ for all $x, y \in \mathcal{X}$.

The *category of reflection spaces* has the class of reflection spaces as objects; a morphism between two objects (\mathcal{X}_1, μ_1) and (\mathcal{X}_2, μ_2) is a map $\varphi : \mathcal{X}_1 \rightarrow \mathcal{X}_2$ such that $\varphi(x \cdot y) = \varphi(x) \cdot \varphi(y)$ for all $x, y \in \mathcal{X}_1$. The *category of symmetric spaces* is the full subcategory whose objects are symmetric spaces.

Remark 2.2. Our definition of a reflection space is taken from [Loo69]. However, Loos defines a symmetric space as a smooth reflection space, in which a local version of (RS4) holds. Our definition of a symmetric space is more demanding, but does not require a topology on \mathcal{X} . An alternative definition of a discrete symmetric space can be found in [Cap05]. In view of (2.1) in Lemma 2.4 below, the definition of a symmetric space given in [Cap05] is equivalent to what we call a reflection space in this article.

Example 2.3.

(i) For any group G , the pair (G, μ_G) with $\mu_G(x, y) := xy^{-1}x$ is a reflection space.

- (ii) For $n \in \mathbb{N}$, the *n -dimensional Euclidean space* \mathbb{E}^n is the symmetric space $(\mathbb{R}^n, \mu_{\mathbb{E}})$ with $\mu_{\mathbb{E}}(x, y) := 2x - y = x - y + x$. Geometrically, $\mu_{\mathbb{E}}(x, \cdot)$ is the point reflection at x .

Note that this example, of course, is just the example of part (i) for the group $(\mathbb{R}^n, +)$.

- (iii) Similar to (ii), spheres and hyperbolic spaces are reflection spaces, where $\mu(x, \cdot)$ is defined as the spherical/hyperbolic point reflection at x .

In view of the previous examples, given a reflection space (\mathcal{X}, μ) the map

$$s_x : \mathcal{X} \rightarrow \mathcal{X}, \quad y \mapsto x \cdot y.$$

is called the *point reflection* at x ; a product of two point reflections is called a *transvection*. By Axiom (RS2) all point reflections are involutions, and Axiom (RS3) states that point reflections (and hence transvections) are automorphisms.

In the sequel denote by $\text{Aut}(\mathcal{X}, \mu)$ the *automorphism group* of \mathcal{X} and by

$$S(\mathcal{X}, \mu) := \{s_x \mid x \in \mathcal{X}\} \subset \text{Aut}(\mathcal{X}, \mu)$$

the subset of all point reflections. The subgroup

$$G(\mathcal{X}, \mu) := \langle S(\mathcal{X}, \mu) \rangle < \text{Aut}(\mathcal{X}, \mu)$$

generated by the set $S(\mathcal{X}, \mu)$ of point reflections is called the *main group* of (\mathcal{X}, μ) , and the subgroup

$$\text{Trans}(\mathcal{X}, \mu) := \langle s_x \circ s_y \mid x, y \in \mathcal{X} \rangle < G(\mathcal{X}, \mu)$$

generated by all transvections is called the *transvection group*. By definition, $\text{Trans}(\mathcal{X}, \mu)$ has index at most 2 in $G(\mathcal{X}, \mu)$. The reflection space (\mathcal{X}, μ) is called *homogeneous* if $\text{Aut}(\mathcal{X}, \mu)$ acts transitively on \mathcal{X} , and *reflection-homogeneous* if $G(\mathcal{X}, \mu)$ acts transitively on \mathcal{X} .

The following formula describes the behavior of point reflections under conjugation.

Lemma 2.4. *Let (\mathcal{X}, μ) be a reflection space, $x, y \in \mathcal{X}$ and $\alpha \in \text{Aut}(\mathcal{X}, \mu)$. Then*

$$\alpha \circ s_y \circ \alpha^{-1} = s_{\alpha(y)}.$$

In particular,

$$s_x \circ s_y \circ s_x = s_{s_x(y)}. \quad (2.1)$$

Proof. For $z \in \mathcal{X}$ one has

$$(\alpha \circ s_y \circ \alpha^{-1})(z) = \alpha(y \cdot \alpha^{-1}z) = \alpha(y) \cdot z = s_{\alpha(y)}(z),$$

which proves the first statement. The second statement then follows from the first and the fact that point reflections are involutive automorphisms. \square

Remark 2.5. The lemma implies that both $G(\mathcal{X}, \mu)$ and $\text{Trans}(\mathcal{X}, \mu)$ are normal in $\text{Aut}(\mathcal{X}, \mu)$. In particular, if one denotes by

$$c_\alpha(g) := \alpha \circ g \circ \alpha^{-1}$$

the conjugation by an element $\alpha \in \text{Aut}(\mathcal{X}, \mu)$, then the assignment $\alpha \mapsto c_\alpha$ induces group homomorphisms

$$c : \text{Aut}(\mathcal{X}, \mu) \rightarrow \text{Aut}(G(\mathcal{X}, \mu)) \quad \text{and} \quad \hat{c} : \text{Aut}(\mathcal{X}, \mu) \rightarrow \text{Aut}(\text{Trans}(\mathcal{X}, \mu)).$$

Note that if $\alpha \in \ker(c)$ then for all $x \in \mathcal{X}$ one has

$$s_{\alpha(x)} = c_\alpha(s_x) = s_x.$$

Thus if \mathcal{X} is symmetric, or more generally $s_x \neq s_y$ for all $x \neq y$ in \mathcal{X} , then

$$c : \text{Aut}(\mathcal{X}, \mu) \rightarrow \text{Aut}(G(\mathcal{X}, \mu))$$

is injective.

2B Involution model and quadratic representation

The following example provides an important construction of reflection spaces. In fact, by Lemma 2.7 below, every symmetric space arises from this construction.

Example 2.6. Let G be a group, let $S \subset G$ be a conjugation-invariant generating subset of involutions, and define a map

$$\psi : S \times S \rightarrow S, \quad \psi(s, r) := s \cdot r = srs.$$

Then (S, ψ) is a reflection space, called the *reflection space associated with the pair (G, S)* . Indeed, for all $x, y \in S$ one has $x \cdot x = xxx = x$ and $x \cdot (x \cdot y) = xxyxx = y$ and, finally,

$$x \cdot (y \cdot z) = xyzyx = xyxxzxyx = xyx \cdot xzx = (x \cdot y) \cdot (x \cdot z),$$

i.e., axioms (RS1), (RS2), (RS3) hold.

The group G acts by automorphisms on (S, ψ) via conjugation and its center $Z(G)$ lies in the kernel of this action. Conversely, any $g \in G$ that acts trivially by conjugation on S necessarily has to be central in G , because S generates G .

One concludes that the main group of (S, ψ) , i.e., the group generated by the point reflections of (S, ψ) , is isomorphic to $G/Z(G)$. Furthermore, (S, ψ) is symmetric if and only if S does not contain any pair of distinct commuting involutions; and it is reflection-homogeneous if and only if S consists of a single conjugacy class in G .

A version of the following lemma has been established in [Cap05] for primitive reflection spaces. Essentially the same proof applies to symmetric spaces.

Lemma 2.7. *Let (\mathcal{X}, μ) be a symmetric space, let $S := S(\mathcal{X}, \mu)$ be the set of its point reflections, and let $G := G(\mathcal{X}, \mu)$ be the main group generated by the point reflections. Then the following assertions hold:*

- (i) $S \subset G$ is a conjugation-invariant subset of G .
- (ii) If (S, ψ) is the reflection space associated with the pair (G, S) , then

$$s : (\mathcal{X}, \mu) \rightarrow (S, \psi), \quad x \mapsto s_x$$

is a G -equivariant isomorphism of reflection spaces.

- (iii) G has trivial center and S does not contain any pair of distinct commuting involutions.
- (iv) \mathcal{X} is reflection-homogeneous if and only if S consists of a single conjugacy class.

Proof. By (2.1) on page 11, the set S is invariant under conjugation by elements in S . Since S generates G , it is therefore invariant under conjugation by elements in G . This shows (i) and makes it meaningful to consider the reflection space (S, ψ) introduced in Example 2.6. Concerning (ii), the map s is surjective by definition, and it is also injective, for, if $s_x = s_y$, then by (RS1) one has $s_x(y) = s_y(y) = y$, which by (RS4) implies $x = y$. By (2.1) the map s is an S -equivariant and hence G -equivariant morphism, proving (ii).

In particular, since G is the main group of (\mathcal{X}, μ) , it is also the main group of (S, ψ) . This, however, implies that G has trivial center by the argument given in Example 2.6. Also, since $(S, \psi) \cong (\mathcal{X}, \mu)$ is symmetric, no two involutions in S commute. This shows (iii). Assertion (iv) follows again from $(S, \psi) \cong (\mathcal{X}, \mu)$. \square

The reflection space (S, ψ) defined in (ii) is referred to as the *involution model* of (\mathcal{X}, μ) . By the lemma, every symmetric space admits an involution model.

Remark 2.8. Rather than realizing a reflection-homogeneous symmetric space (\mathcal{X}, μ) by a suitable generating conjugacy class of involutions of its main group, one can also realize it as a suitable subset of its transvection group.

This embedding, which depends on a choice of basepoint $o \in \mathcal{X}$, is referred to as the *quadratic representation* of \mathcal{X} in [Loo69, Section II.1] (see also [Cap05, Lemma 2.3]). Given $x \in \mathcal{X}$ one defines $t_x := s_x \circ s_o \in \text{Trans}(\mathcal{X}, \mu)$ and sets $T(\mathcal{X}, \mu, o) := \{t_x \mid x \in \mathcal{X}\}$. Then the map

$$t : \mathcal{X} \rightarrow T(\mathcal{X}, \mu, o), \quad x \mapsto t_x$$

is a bijection; indeed, injectivity follows from $t_x \circ s_o = s_x$. This bijection induces on $T(\mathcal{X}, \mu, o)$ the structure of a symmetric space. Now by (2.1) on page 11 for all $x, y \in \mathcal{X}$ one has

$$t_x t_y^{-1} t_x = s_x \circ s_o \circ (s_y \circ s_o)^{-1} s_x \circ s_o = s_x \circ s_y \circ s_x \circ s_o = s_{s_x(y)} \circ s_o = t_{s_x(y)},$$

whence the induced multiplication in this model is given by

$$T(\mathcal{X}, \mu, o) \times T(\mathcal{X}, \mu, o) \rightarrow T(\mathcal{X}, \mu, o), \quad (s, t) \mapsto s \cdot t = st^{-1}s. \quad (2.2)$$

Note that $T(\mathcal{X}, \mu, o)$ is a reflection subspace of the group $\text{Trans}(\mathcal{X}, \mu)$, where the latter is equipped with its canonical reflection space structure as given by Example 2.3(i).

As another consequence of (2.1) observe that for all $x, y \in \mathcal{X}$ one has

$$s_x \circ s_y = s_x \circ s_o \circ (s_o \circ s_y \circ s_o) \circ s_o = s_x \circ s_o \circ s_{s_o(y)} \circ s_o = t_x \circ t_{s_o(y)}.$$

In particular, $T(\mathcal{X}, \mu, o)$ actually generates the transvection group.

2C Topological reflection spaces

All the concepts introduced in the previous subsection make sense in a topological setting.

Definition 2.9. Let \mathcal{X} be a topological space and let $\mu : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}$, $(x, y) \mapsto x \cdot y$ be a continuous map.

- (i) (\mathcal{X}, μ) is called a *topological reflection space* if it satisfies axioms (RS1)–(RS3), and a *topological symmetric space* if it satisfies axioms (RS1)–(RS4).
- (ii) The *categories of topological reflection spaces* and *of topological symmetric spaces* are defined by requiring morphisms to be continuous in addition to preserving the product.
- (iii) The *automorphism group* $\text{Aut}(\mathcal{X}, \mu)$, the *main group* $G(\mathcal{X}, \mu)$ and the *transvection group* $\text{Trans}(\mathcal{X}, \mu)$ are defined as in the abstract setting with the additional requirement that automorphisms be homeomorphisms.

The following topological variants of Examples 2.3 and 2.6 provide examples for topological reflection spaces.

Example 2.10.

- For any topological group G , the pair (G, μ_G) with $\mu_G(x, y) := xy^{-1}x$ is a topological reflection space.
- The n -dimensional Euclidean space $\mathbb{E}^n = (\mathbb{R}^n, \mu_{\mathbb{E}})$ is a topological symmetric space with its canonical vector space topology. Similarly, spheres and hyperbolic spaces are topological reflection spaces with their standard topologies.
- Given a topological group G and a conjugation-invariant generating subset S of involutions, then S is a topological reflection space with respect to the multiplication $r \cdot s = rsr$.

Remark 2.11. We emphasize that Lemma 2.7 does not have counterpart in the setting of general topological reflection spaces. More precisely, if (\mathcal{X}, μ) is a topological symmetric space, then the abstract reflection space underlying (\mathcal{X}, μ) can of course be realized as a subset of its main group (or inside its transvection group), but finding a group topology on either of these groups which restricts to the given topology on (\mathcal{X}, μ) is difficult without additional hypotheses on the structure of the topological symmetric space.

2D Flats in topological reflection spaces

Throughout this section let (\mathcal{X}, μ) be a topological reflection space and let $x, y, z \in \mathcal{X}$. Since point reflections are involutions one has $s_x(y) = z$ if and only if $s_x(z) = y$. In this situation one calls x a *midpoint* of y and z .

In [LL07] the authors develop a rich structure theory of reflection spaces in which any pair of points has a unique midpoint, see [LL07, Section 2, Axiom (S4)]. We will see in Corollary 5.11 that every non-spherical Kac–Moody symmetric space contains pairs of points that do not admit a midpoint, hence it is important for us to develop the basic theory of reflection spaces without assuming the existence of midpoints. Note also that in general topological reflection spaces midpoints, if they exist, need not be unique, as is already clear from the example of spheres.

Definition 2.12. Let (\mathcal{X}, μ) be a topological reflection space and $\mathcal{Y} \subseteq \mathcal{X}$ a subspace.

- (i) $\mathcal{Y} \subseteq \mathcal{X}$ is a *reflection subspace* if for $p, q \in \mathcal{Y}$ also $s_p(q) \in \mathcal{Y}$.
- (ii) $\mathcal{Y} \subseteq \mathcal{X}$ is *midpoint convex* if for all $p, q \in \mathcal{Y}$ there is a midpoint of p and q in \mathcal{Y} .

Note that a reflection subspace of a topological reflection space (\mathcal{X}, μ) is itself a topological reflection space with respect to the restriction of μ and the subspace topology. Also note that the closure of a reflection subspace \mathcal{Y} is a reflection subspace², whereas generally it is unclear to us whether the closure of a midpoint convex subset is midpoint convex, if \mathcal{X} is not locally compact.³

Example 2.13. The n -dimensional Euclidean space \mathbb{E}^n is midpoint convex. Moreover, $(\mathbb{Z}^n, \mu_{\mathbb{E}})$ is a reflection subspace of \mathbb{E}^n which is not midpoint convex, whereas $(\mathbb{Q}^n, \mu_{\mathbb{E}})$ is a midpoint convex reflection subspace of \mathbb{E}^n , albeit not closed. The closed midpoint convex reflection subspaces of \mathbb{E}^n are exactly the affine subspaces, i.e., the translates of \mathbb{R} -vector subspaces of the underlying \mathbb{R}^n .

Definition 2.14. Let (\mathcal{X}, μ) be a topological reflection space and $F \subseteq \mathcal{X}$ a reflection subspace.

- (i) $x, y \in \mathcal{X}$ *weakly commute* if for all $z \in \mathcal{X}$ one has

$$x \cdot (z \cdot (y \cdot z)) = y \cdot (z \cdot (x \cdot z)).$$

- (ii) $x, y \in \mathcal{X}$ *commute* if for all $a, b \in \mathcal{X}$ one has

$$x \cdot (a \cdot (y \cdot b)) = y \cdot (a \cdot (x \cdot b)).$$

- (iii) F is *(weakly) abelian* if all its points (weakly) commute.
- (iv) F is called a *(weak) flat* if it is closed, midpoint convex, (weakly) abelian, and contains at least two points.

²For, if $x, y \in \overline{\mathcal{Y}}$, then there exist nets $(x_\alpha), (y_\alpha)$ in \mathcal{Y} converging to x and y respectively, whence $x \cdot y = \lim x_\alpha \cdot y_\alpha \in \overline{\mathcal{Y}}$ by joint continuity of multiplication.

³In case \mathcal{X} actually is locally compact, one can argue as follows. Let \mathcal{Y} be a midpoint convex subset of \mathcal{X} and let $\overline{\mathcal{Y}}$ be its closure in \mathcal{X} . Then \mathcal{Y} contains nets (x_α) converging to x and (y_α) converging to y . By local compactness the net (z_α) consisting of the midpoints z_α of x_α and y_α contains a subnet that in $\overline{\mathcal{Y}}$ converges to some point z . By continuity, the reflection s_z interchanges x and y , i.e., $z \in \overline{\mathcal{Y}}$ is a midpoint of x and y .

(v) F is called a *Euclidean flat* of *rank* n if it is closed and isomorphic to \mathbb{E}^n as a topological reflection space.

Lemma 2.15. *Let (\mathcal{X}, μ) be a reflection space.*

- (i) *Every Euclidean flat is a flat, and every flat is a weak flat.*
- (ii) *Every $g \in \text{Aut}(\mathcal{X}, \mu)$ preserves the collection of weak flats, and the subcollections of flats, Euclidean flats and Euclidean flats of a given rank n .*
- (iii) *Every weak subflat of a Euclidean flat is Euclidean.*

Proof. The first statement of (i) is contained in [Loo69, Proposition III.2.5], and the second statement of (i) is obvious, (ii) is immediate from the definitions, and (iii) follows from Example 2.13. \square

For an illustration that Euclidean flats are weakly abelian see Figure 1.

Remark 2.16. Theorem 5.15 below states that in Kac–Moody symmetric spaces every weak flat is Euclidean, whence all three notions of flats coincide in that situation.

The notion of an abelian reflection subspace is taken from [Loo69, III.2.2, p. 134ff]. Note that spheres and hyperbolic spaces are not weakly abelian, thus among constant curvature smooth examples, being weakly abelian is equivalent to flatness in the sense of zero curvature. In the smooth homogeneous context, being abelian is equivalent to the vanishing of the curvature tensor by [Loo69, Proposition III.2.5].

Assume now that \mathcal{X} is a topological reflection space and that $F \subset \mathcal{X}$ is a Euclidean flat of rank n . If one fixes a topological isomorphism $\varphi : \mathbb{E}^n \rightarrow F$, then every automorphism $\alpha \in \text{Aut}(\mathcal{X})$ that stabilizes the set F induces a map $\hat{\alpha} := \varphi \circ \alpha \circ \varphi^{-1} : \mathbb{E}^n \rightarrow \mathbb{E}^n$.

Proposition 2.17. *If $\alpha \in \text{Aut}(\mathcal{X})$ preserves F , then $\hat{\alpha} := \varphi \circ \alpha \circ \varphi^{-1}$ is an affine transformation, i.e., $\hat{\alpha}$ is linear-by-translation.*

Proof. The map $\hat{\alpha} : \mathbb{E}^n \rightarrow \mathbb{E}^n$ is a topological isomorphism of reflection spaces. In particular, for all $x, y \in \mathbb{E}^n$ one has

$$\hat{\alpha}(2x - y) = \hat{\alpha}(\mu(x, y)) = \mu(\hat{\alpha}(x), \hat{\alpha}(y)) = 2\hat{\alpha}(x) - \hat{\alpha}(y).$$

The group of translations acts transitively on \mathbb{E}^n , so up to composition of $\hat{\alpha}$ with an appropriate translation one may assume $\hat{\alpha}(0) = 0$. By setting $y = 0$ one then concludes that $\hat{\alpha}$ is homogeneous with respect to powers of 2 and by setting $x = 0$ one concludes that $\hat{\alpha}$ is homogeneous with respect to -1 . Replacing x by $\frac{1}{2}x$ and y by $-y$ then implies that $\hat{\alpha}$ is additive. Since $\mathbb{Z}[\frac{1}{2}]$ is dense in \mathbb{R} , this implies \mathbb{R} -linearity of $\hat{\alpha}$. \square

By abuse of language one says that α *acts affine-linearly* on F .

2E Geodesics and translation groups

In this section we prove Theorem 1.3.

Definition 2.18. Let (\mathcal{X}, μ) be a topological reflection space. A Euclidean flat $\gamma \subset \mathcal{X}$ of rank 1 is called a *geodesic*, and the subset

$$T_\gamma := \{s_p \circ s_q \mid p, q \in \gamma\} \subset \text{Trans}(\mathcal{X}, \mu).$$

is called the associated *translation group*.

It is not obvious a priori that T_γ is a group. However, one can show the following:

Proposition 2.19. *Let (\mathcal{X}, μ) be a topological reflection space and $\gamma \subset \mathcal{X}$ a geodesic.*

- (i) $T_\gamma \cong (\mathbb{R}, +)$ is a one-parameter subgroup of $\text{Trans}(\mathcal{X}, \mu)$.
- (ii) T_γ acts sharply transitively on γ by Euclidean translations.
- (iii) If $t_1, t_2 \in T_\gamma$ and $t_1|_\gamma = t_2|_\gamma$, then $t_1 = t_2$.

For the proof of Proposition 2.19 use the following notation: Fix a *parametrization* of γ , i.e., a bijection $\varphi : \mathbb{R} \rightarrow \gamma$ such that $\varphi(2x - y) = s_{\varphi(x)}(\varphi(y))$, given $x \in \mathbb{R}$ abbreviate $s_x := s_{\varphi(x)}$, and given $x, y \in \mathbb{R}$ define a transvection

$$t_\gamma[x, y] := t[x, y] := s_y \circ s_{(x+y)/2}. \quad (2.3)$$

By construction, $t[x, y]$ is a transvection along γ which maps $\varphi(x)$ to $\varphi(y)$, hence the notation. Note that the restriction of this transvection to γ corresponds via φ to the translation by $y - x$ in \mathbb{R} . With this notation one has $T_\gamma = \{t_\gamma[x, y] \mid x, y \in \mathbb{R}\}$ and, thus, Proposition 2.19 is a consequence of the following lemma:

Lemma 2.20. *With the notation just introduced the following hold.*

- (i) For every $x \in \mathbb{R}$ the map $t_{\gamma, x} : (\mathbb{R}, +) \rightarrow T_\gamma, y \mapsto t[x, x + y]$ is an injective group homomorphism.
- (ii) For all $x, y \in \mathbb{R}$ one has $t[x, x + y] = t[0, y]$. In particular, $t_{\gamma, x}$ is onto for every $x \in \mathbb{R}$.

Proof. (i) By Lemma 2.4 and the formula for Euclidean reflections in \mathbb{R} one has

$$s_x \circ s_y \circ s_x = s_{2x-y} \quad (x, y \in \mathbb{R}). \quad (2.4)$$

This allows one to rewrite (2.3) as

$$t[x, y] = s_y \circ s_{(x+y)/2} = s_{(x+y)/2} \circ (s_{(x+y)/2} \circ s_y \circ s_{(x+y)/2}) = s_{(x+y)/2} \circ s_x, \quad (2.5)$$

which yields in particular $t[x, x + y] = s_{x+y/2} \circ s_x$ and, thus,

$$t[x, x + y] \circ t[x, x - y] = s_{x+y/2} \circ (s_x \circ s_{x-y/2} \circ s_x) = s_{x+y/2} \circ s_{x+y/2} = \text{Id}. \quad (2.6)$$

It also follows from (2.4) and (2.5) that

$$t[x, x + y/2]^2 = (s_{x+y/4} \circ s_x \circ s_{x+y/4}) \circ s_x = s_{x+y/2} \circ s_x = t[x, x + y],$$

whence by induction

$$t[x, x + y] = t[x, x + 2^{-n}y]^{2^n}. \quad (2.7)$$

Next one observes that

$$(s_{x+y/2} \circ s_x)(\varphi(x + y/2)) = s_{x+y/2}(\varphi(x - y/2)) = \varphi(x + 3y/2),$$

and inductively one obtains for all $k \in \mathbb{N}$,

$$(s_{x+y/2} \circ s_x)^k(\varphi(x + y/2)) = \varphi(x + (2k + 1)y/2)$$

Thus, if $n = 2k + 1$ is an odd positive integer, then

$$\begin{aligned} t[x, x + y]^n &= (s_{x+y/2} \circ s_x)^{2k+1} = (s_{x+y/2} \circ s_x)^k \circ s_{x+y/2} \circ (s_x \circ s_{x+y/2})^k \circ s_x \\ &= s_{x+(2k+1)y/2} \circ s_x = t[x, x + ny]. \end{aligned}$$

Combining this with (2.6) and (2.7) it follows that $t_{\gamma, x}|_{\mathbb{Q}}$ is a homomorphism, whence continuity of μ implies that $t_{\gamma, x}$ is a homomorphism. Since $t[x, x + y](x) = \varphi(x + y)$ it is injective.

(ii) Let $x, y \in \mathbb{R}$ and set $z := x/2 + y/4$ so that

$$s_z(\varphi(0)) = \varphi(x + y/2), \quad s_z(\varphi(x)) = \varphi(y/2).$$

It follows from (i) that $s_0 \circ s_x$ and $s_z \circ s_x$ commute and hence

$$s_0 \circ s_x = (s_z \circ s_x) \circ (s_0 \circ s_x) \circ (s_z \circ s_x)^{-1} = s_z \circ s_x \circ s_0 \circ s_z = s_{y/2} \circ s_{x+y/2}.$$

One deduces that $s_{y/2} \circ s_0 = s_{x+y/2} \circ s_x$, and thus taking inverses

$$t[0, y] = s_{y/2} \circ s_0 = s_{x+y/2} \circ s_x = t[x, x + y]. \quad \square$$

Remark 2.21. Proposition 2.19 generalizes [Lan99, Proposition XIII.5.5] as well as [Nee02, Theorem 3.6(iv)] to arbitrary topological reflection spaces: any geodesic in any topological reflection space defines a one-parameter subgroup of its automorphism group. It is quite remarkable that this property relies purely on group theory and elementary Euclidean geometry and does not require any differentiable structure whatsoever.

2F Geodesically connected reflection spaces

Definition 2.22. Let (\mathcal{X}, μ) be a topological reflection space and $\gamma \subset \mathcal{X}$ a geodesic. A compact connected subset $\sigma \subset \gamma$ with non-empty relative interior is called a *geodesic segment*. A triple $\vec{\sigma} = (\sigma, s(\vec{\sigma}), t(\vec{\sigma}))$, where σ is a geodesic segment and $s(\vec{\sigma})$ and $t(\vec{\sigma})$ are the endpoints of σ is called an *oriented geodesic segment* from $s(\vec{\sigma})$ to $t(\vec{\sigma})$. Given an oriented geodesic segment $\vec{\sigma}$ in γ the *parallel transport along $\vec{\sigma}$* is defined as the unique transvection $t[\vec{\sigma}] \in T_\gamma$ mapping $s(\vec{\sigma})$ to $t(\vec{\sigma})$.

An *oriented piecewise geodesic curve* is a sequence $\vec{\sigma} = (\vec{\sigma}_1, \vec{\sigma}_2, \dots, \vec{\sigma}_n)$ of oriented geodesic segments with $t(\sigma_i) = s(\sigma_{i+1})$. Then set $s(\vec{\sigma}) := s(\vec{\sigma}_1)$ and $t(\vec{\sigma}) := t(\vec{\sigma}_n)$ and say that $\vec{\sigma}$ is a curve from $s(\vec{\sigma})$ to $t(\vec{\sigma})$. Also define *parallel transport along $\vec{\sigma}$* as the transvection

$$t[\vec{\sigma}] := t[\vec{\sigma}_n] \circ \dots \circ t[\vec{\sigma}_2] \circ t[\vec{\sigma}_1].$$

(\mathcal{X}, μ) is *geodesically connected* if for all $p, q \in \mathcal{X}$ there exists an oriented piecewise geodesic curve from p to q .

Recall that in a finite-dimensional Riemannian symmetric space any pair of points lies on a common geodesic. This is no longer the case for Kac–Moody symmetric spaces by Corollary 5.11 below. Nevertheless, Kac–Moody symmetric spaces still satisfy the weaker property of being geodesically connected by Lemma 5.13; as it turns out this is enough to deduce various basic structural features such as the following information concerning the transvection group:

Proposition 2.23. *Let (\mathcal{X}, μ) be a geodesically connected topological reflection space.*

- (i) *Trans (\mathcal{X}, μ) acts transitively on \mathcal{X} . In particular, \mathcal{X} is reflection-homogeneous.*
- (ii) *Trans (\mathcal{X}, μ) is generated by the one-parameter subgroups T_γ , where γ runs through all geodesics in \mathcal{X} .*

Proof. (i) If p, q are distinct points in \mathcal{X} and $\vec{\sigma}$ is an oriented piecewise geodesic curve from p to q , then $t[\vec{\sigma}] \in \text{Trans}(\mathcal{X}, \mu)$ maps p to q .

- (ii) Let p and q be distinct points in \mathcal{X} and let $\vec{\sigma} = (\vec{\sigma}_1, \vec{\sigma}_2, \dots, \vec{\sigma}_n)$ be a piecewise oriented geodesic curve between p and q . It suffices to show that $s_q \circ s_p \in \text{Trans}(\mathcal{X}, \mu)$ can be written as a product of elements of the translation groups corresponding to the geodesics involved in the above curve. To this end set $p_i = t(\vec{\sigma}_i)$ and $p_0 := p$, $q_i := s_{p_i}(p_{i-1})$. Then

$t_i := s_{q_i} \circ s_{p_i} \in T_{\gamma_i}$ where γ_i is the geodesic containing $\vec{\sigma}_i$ and $t_i(p_{i-1}) = q_i$. Thus (ii) follows from the computation

$$\begin{aligned} s_q \circ s_p &= s_{p_n} \circ s_{p_0} = (s_{p_n} \circ s_{p_{n-1}}) \circ (s_{p_{n-1}} \circ s_{p_{n-2}}) \cdots (s_{p_1} \circ s_{p_0}) \\ &= ((s_{p_n} \circ s_{p_{n-1}} \circ s_{p_n}) \circ s_{p_n}) \circ \cdots \circ ((s_{p_1} \circ s_{p_0} \circ s_{p_1}) \circ s_{p_1}) \\ &= (s_{q_n} \circ s_{p_n}) \circ \cdots \circ (s_{q_1} \circ s_{p_1}) \\ &= t_n \circ \cdots \circ t_1. \end{aligned}$$

□

Remark 2.24. Part (ii) of the proposition provides an obstruction for a group to occur as the transvection group of some geodesically connected topological reflection space: Any such group has to be generated by a family of subgroups isomorphic to $(\mathbb{R}, +)$.

2G Local automorphisms of strongly transitive reflection spaces

In a general topological reflection space, it is unclear to us whether every flat is contained in a maximal flat. Indeed, while every midpoint convex abelian reflection subspace certainly is contained in a maximal midpoint convex abelian reflection subspace, there is no reason for this maximal space to be closed. As it is unclear to us whether closures of midpoint convex subsets are again midpoint convex, we are unable to guarantee even the existence of a single maximal flat in this generality. However, if maximal flats exist, then they often give a major insight into the structure of the topological reflection space, since every automorphism has to preserve maximal flats and their intersection patterns.

Definition 2.25. Let \mathcal{X} be a topological reflection space which admits maximal flats.

- (i) A pair (p, F) where F is a maximal flat in \mathcal{X} and $p \in F$ is a point is called a *pointed maximal flat*.
- (ii) Let G be a group acting on \mathcal{X} by automorphisms. We say that the action is *strongly transitive* if G acts transitively on pointed maximal flats.
- (iii) \mathcal{X} is called *strongly transitive* if $\text{Aut}(\mathcal{X})$ acts strongly transitively on \mathcal{X} .

The following observation is often useful for checking strong transitivity. It will be used, for instance, in Corollary 5.16 below in order to show that Kac–Moody symmetric spaces are strongly transitive.

Proposition 2.26. Let $\text{Trans}(\mathcal{X}) < G < \text{Aut}(\mathcal{X})$. If G acts transitively on maximal flats in \mathcal{X} and if one, whence all, of these are Euclidean, then G acts strongly transitively on \mathcal{X} .

Proof. Any maximal flat $F \subset \mathcal{X}$ by definition is a reflection subspace and, hence, each point reflection of F is induced by a point reflection of \mathcal{X} . Since G acts transitively on the set of maximal flats of \mathcal{X} , the flat F is Euclidean, i.e., $F \cong \mathbb{E}^n$ for some n . It follows that the stabilizers of F in $\text{Trans}(\mathcal{X})$ and, thus, in G contain $\text{Trans}(F)$, and hence act transitively on F . This implies the proposition. □

Remark 2.27. If in the situation of the preceding proposition the maximal Euclidean flats of \mathcal{X} have rank k , then what we call strong transitivity in the present article coincides with the notion of *k-flat homogeneity* in the literature.

Let \mathcal{X} be a topological reflection space which contains a maximal flat, which moreover is Euclidean and let G be a group with $\text{Trans}(\mathcal{X}) < G < \text{Aut}(\mathcal{X})$. Moreover, assume that G acts transitively on maximal flats of \mathcal{X} , and let (p, F) be a pointed maximal flat in \mathcal{X} . By Proposition 2.26, G acts strongly transitively on \mathcal{X} and F is Euclidean. Denote by

$$\text{Stab}_G(p, F) := \{g \in G \mid g.F = F, g.p = p\} \quad \text{and} \quad \text{Fix}_G(p, F) := \{g \in G \mid \forall f \in F : g.f = f\}$$

the stabilizer, respectively fixator of (p, F) in G .

Definition 2.28. Let \mathcal{X} be a topological reflection space which contains a maximal flat F , which moreover is Euclidean, and let $p \in F$. A point $q \in F$ is called *singular with respect to p* if there exists a second maximal flat distinct from F containing both p and q , and *regular with respect to p* otherwise. Denote by $F^{\text{reg}}(p) \subset F$ the subset of regular points in F with respect to p , and by $F^{\text{sing}}(p) \subset F$ the subset of singular points in F with respect to p .

A map $f : F \rightarrow F$ is called a *local automorphism* of (p, F) , if in some (hence any) linear parametrization of F with origin p it is given by a linear map and moreover preserves the decomposition

$$F = F^{\text{reg}}(p) \sqcup F^{\text{sing}}(p).$$

Denote by $\text{Aut}(p, F)$ the group of local automorphisms of (F, p) .

Proposition 2.29. Let \mathcal{X} be a topological reflection space which contains a maximal flat, which moreover is Euclidean, and assume moreover that G acts transitively on maximal flats of \mathcal{X} .

- (i) The group $W(G \curvearrowright \mathcal{X}) := \text{Stab}_G(p, F) / \text{Fix}_G(p, F)$ is independent of the choice of pointed flat (p, F) up to isomorphism.
- (ii) There is a homomorphism $\rho_F : W(G \curvearrowright \mathcal{X}) \rightarrow \text{Aut}(p, F)$, $\rho_F([f]) := f|_F$, which is independent of the choice of pointed flat (p, F) up to isomorphism.

Proof. By Proposition 2.26 the group G acts strongly transitively on \mathcal{X} . Assertion (i) and the second statement of assertion (ii) are immediate from strong transitivity. The first statement of assertion (ii) follows from Proposition 2.17 and Lemma 2.15. \square

Definition 2.30. (i) The group $W(G \curvearrowright \mathcal{X})$ is called the *(geometric) Weyl group* of the action $G \curvearrowright \mathcal{X}$.

- (ii) The homomorphism $\rho_F : W(G \curvearrowright \mathcal{X}) \rightarrow \text{Aut}(p, F)$ as the *local action* of G on \mathcal{X} .

Note that, if \mathcal{X} is a Riemannian symmetric space, then the local action of $\text{Aut}(\mathcal{X})$ is surjective (i.e., every local automorphism globalizes) if and only if the automorphism group of the underlying Dynkin diagram coincides with the automorphism group of the corresponding Coxeter diagram. Theorem 6.10 below states that the same holds also for Kac–Moody symmetric spaces.

3 Split real Kac–Moody groups and their Iwasawa decompositions

3A Groups with RGD systems

This subsection provides some necessary background concerning groups with RGD systems (see [AB08, Sections 8.5]); for the definitions of a prenilpotent pair of roots as well as the definitions of the “closed” interval $[\alpha, \beta]$ and the “open” interval $] \alpha, \beta[$ of roots α, β used therein see [AB08, Sections 8.5.2, 8.5.3].

Definition 3.1. Let (W, S) be a Coxeter system with root system Φ and let Φ^+ be a subset of positive roots. A *centered RGD system* is a triple $(G, \{U_\alpha\}_{\alpha \in \Phi}, T)$, where G is a group, $T < G$ a subgroup and $\{U_\alpha\}_{\alpha \in \Phi}$ is a family of subgroups of G subject to the following axioms:

- (RGD1) For each root $\alpha \in \Phi$, one has $U_\alpha \neq \{1\}$.
- (RGD2) For each prenilpotent pair $\{\alpha, \beta\} \subseteq \Phi$ of distinct roots, one has $[U_\alpha, U_\beta] \subseteq \langle U_\gamma \mid \gamma \in] \alpha, \beta[\rangle$.
- (RGD3) For each $s \in S$ there exists a function $\mu_s : U_{\alpha_s} \setminus \{1\} \rightarrow G$ such that for all $u \in U_{\alpha_s} \setminus \{1\}$ and $\alpha \in \Phi$ one has $\mu_s(u) \in U_{-\alpha_s} u U_{-\alpha_s}$ and $\mu_s(u) U_\alpha \mu_s(u)^{-1} = U_{s(\alpha)}$.

(RGD4) For each $s \in S$ one has $U_{-\alpha_s} \not\subseteq U_+ := \langle U_\alpha \mid \alpha \in \Phi^+ \rangle$.

(RGD5) $G = \langle U_\alpha \mid \alpha \in \Phi \rangle$.

(RGD6) The group T normalizes every U_α .

Every centered RGD system $(G, \{U_\alpha\}_{\alpha \in \Phi}, T)$ gives rise to a *saturated twin BN pair* (B_+, B_-, N) in the sense of Tits as follows (cf. [AB08, Theorem 8.80]). If $\mu_s : U_{\alpha_s} \setminus \{1\} \rightarrow U_{-\alpha_s} U_{\alpha_s} U_{-\alpha_s}$ is the map provided by (RGD3), the group U_+ is as in (RGD4) and $U_- := \langle U_\alpha \mid \alpha \in -\Phi \rangle$, then T normalizes both U_+ and U_- and one obtains a twin BN-pair (B_+, B_-, N) by

$$\begin{aligned} N &:= T \cdot \langle \mu_s(u) \mid u \in U_\alpha \setminus \{1\}, s \in S \rangle, \\ B_+ &:= T \ltimes U_+, \\ B_- &:= T \ltimes U_-. \end{aligned}$$

This twin BN-pair satisfies the saturation property $B_+ \cap B_- = T$ (cf. [AB08, Corollary 8.78]) and $T = \bigcap_{\alpha \in \Phi} N_G(U_\alpha)$ (cf. [AB08, Corollary 8.79]); note that $N = N_G(T)$ by [AB08, Theorem 6.87(2) and Theorem 8.80].

The twin BN-pair (B_+, B_-, N) then gives rise to two buildings with respective chamber sets $\Delta_\pm := G/B_\pm$ and a twinning between them ([AB08, Section 8.9]), which leads to a twin building.

The theory of twin buildings is an invaluable tool for studying groups with an RGD-system. Refer to [AB08, Section 6.3 and Chapter 8] for general background information on twin buildings endowed with a group action and to [HKM13] for a setup of twin buildings that has been specifically tailored to suit the properties of topological Kac–Moody groups.

3B Complex and split real topological Kac–Moody groups

Definition 3.2. A *generalized Cartan matrix* is an integral square matrix $\mathbf{A} = (a_{ij})_{1 \leq i, j \leq n} \in M_n(\mathbb{Z})$ satisfying $a_{ii} = 2$, $a_{ij} \leq 0$ for $i \neq j$, and $a_{ij} = 0$ if and only if $a_{ji} = 0$. (Cf. [Kac90, §1.1].)

The *Dynkin diagram* $\Gamma_{\mathbf{A}}$ of \mathbf{A} is the edge-labelled graph with vertex set $\mathcal{V} = \{1, \dots, n\}$ and edge set $\mathcal{E} := \{\{i, j\} \subset \mathcal{V} \mid i \neq j, a_{ij}a_{ji} \neq 0\}$. If $e \in \mathcal{E}$ joins the vertices i and j and $a_{ij} > a_{ji}$, then e is labelled by the number $a_{ij}a_{ji}$, by an arrow from i to j and, if $a_{ij}a_{ji}$ is not prime, by the values a_{ij} and a_{ji} . The matrix \mathbf{A} and the diagram $\Gamma_{\mathbf{A}}$ are called *irreducible* if $\Gamma_{\mathbf{A}}$ is connected, *two-spherical* if $\Gamma_{\mathbf{A}}$ has no labels $a_{ij}a_{ji} > 3$, *spherical* if \mathbf{A} is the Cartan matrix of a finite-dimensional Lie group, and *non-spherical* otherwise. The *Coxeter diagram* is induced by the Dynkin diagram $\Gamma_{\mathbf{A}}$ by removing all arrows and all values a_{ij} and a_{ji} and replacing labels equal to one by three, labels equal to two by four, labels equal to three by six, and labels greater than three by ∞ (see also Section AC).

The generalized Cartan matrix \mathbf{A} is called *symmetrizable* if there exist a symmetric matrix $B = (b_{ij}) \in M_n(\mathbb{R})$ and diagonal matrix $D = \text{diag}(\varepsilon_1, \dots, \varepsilon_n) \in M_n(\mathbb{R})$ with $\varepsilon_j > 0$ such that $\mathbf{A} = DB$. The matrix D is not unique, but one can choose D to be *minimal* in the sense of [Kum02, Definition 1.5.1]: Each ε_i is a positive integer, and if $\text{diag}(\varepsilon'_1, \dots, \varepsilon'_n)$ is another such matrix, then $\varepsilon_i \leq \varepsilon'_i$ for all i .

The key results of this article concerning Kac–Moody symmetric spaces hold in the presence of the following general hypotheses.

Convention 3.3. In this article $\mathbf{A} \in M_n(\mathbb{Z})$ denotes a non-spherical non-affine irreducible symmetrizable generalized Cartan matrix.

A generalized Cartan matrix \mathbf{A} is the key ingredient for defining a topological split Kac–Moody group over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Assume first that \mathbf{A} is two-spherical. Under this additional assumption there is a very efficient way of defining these groups as colimits of diagrams of groups as described in [AM97]: For each vertex $i \in \mathcal{V}$ of the Dynkin diagram $\Gamma_{\mathbf{A}}$ define $G_i(\mathbb{K}) := \text{SL}_2(\mathbb{K})$.

For every pair $\{i, j\} \subset \mathcal{V}$ ($i \neq j$) define $G_{\{i, j\}}(\mathbb{K})$ as the split Lie group over \mathbb{K} of rank two whose Dynkin diagram is the full labelled subgraph of $\Gamma_{\mathbf{A}}$ on vertices i, j . A fixed choice of a root basis provides natural inclusion maps $\iota_i : G_i(\mathbb{K}) \hookrightarrow G_{\{i, j\}}(\mathbb{K})$.

Consider the amalgam $\mathcal{A}_{\mathbf{A}}$ of topological groups formed by the Lie groups $G_i(\mathbb{K})$, $i \in \mathcal{V}$, and $G_{\{i, j\}}(\mathbb{K})$, $i \neq j$, together with the canonical inclusions. The colimit of this amalgam in the category of topological groups turns out to be a Hausdorff topological group $G_{\mathbb{K}}(\mathbf{A})$, which is moreover a k_{ω} space in the sense of Definition 3.25 below (see [HKM13, Theorem 7.22]). This colimit is abstractly isomorphic to the quotient of the free group generated by the elements of the groups $G_i(\mathbb{K})$ modulo the relations given as products of conjugates of the relations contained in $G_{\{i, j\}}(\mathbb{K})$; its topology equals the finest group topology such that the natural embeddings of the Lie groups $G_i(\mathbb{K})$ are continuous.

Definition 3.4. The group $G_{\mathbb{R}}(\mathbf{A})$ (respectively $G_{\mathbb{C}}(\mathbf{A})$) is called the *simply connected centered split real (resp. complex) Kac–Moody group* of type \mathbf{A} . The topology on $G_{\mathbb{K}}(\mathbf{A})$ defined above is called the *Kac–Peterson topology*.

Given a subset $I \subset \mathcal{V}$ the subgroup $G_I(\mathbb{K}) := \langle G_i(\mathbb{K}) \mid i \in I \rangle$ is called a *standard rank $|I|$ subgroup* of $G_{\mathbb{K}}(\mathbf{A})$. Denote by $\varphi_I : G_I(\mathbb{K}) \rightarrow G_{\mathbb{K}}(\mathbf{A})$ the canonical inclusion; if $|I| = 1$ one simply writes φ_i and G_i instead of $\varphi_{\{i\}}$ and $G_{\{i\}}$ respectively.

The embedding $\mathbb{R} \hookrightarrow \mathbb{C}$ induces embeddings $G_i(\mathbb{R}) \hookrightarrow G_i(\mathbb{C})$ and $G_{\{i, j\}}(\mathbb{R}) \hookrightarrow G_{\{i, j\}}(\mathbb{C})$ and hence an embedding $G_{\mathbb{R}}(\mathbf{A}) \hookrightarrow G_{\mathbb{C}}(\mathbf{A})$. Since our main focus lies on the real case, we will subsequently write $G := G_{\mathbb{R}}(\mathbf{A})$, $G_i := G_i(\mathbb{R})$, etc.

The topological Kac–Moody groups $G_{\mathbb{R}}(\mathbf{A})$ and $G_{\mathbb{C}}(\mathbf{A})$ and all of the notions pertaining to these groups as defined in this subsection can also be defined without the assumption that \mathbf{A} be two-spherical, and the results in this article are valid without the assumption of two-sphericity unless explicitly stated otherwise. However, in this more general setting the amalgamation results from [AM97] are not available, and thus the definitions become substantially more technical. We refer the reader to [Cap09], [HKM13, Chapter 7], [R  m02], [Tit87] for the general definitions.

3C The adjoint quotient and the semisimple adjoint quotient

The group $G = G_{\mathbb{R}}(\mathbf{A})$ can be considered as an infinite-dimensional generalization of a finite-dimensional semisimple split real Lie group. In fact, if \mathbf{A} is a spherical irreducible (generalized) Cartan matrix, then the resulting Kac–Moody group G is an algebraically simply connected simple split real Lie group. In particular, the center of G is 0-dimensional. In this case \mathbf{A} is automatically symmetrizable and, in fact, invertible.

A non-spherical irreducible symmetrizable generalized Cartan matrix \mathbf{A} on the other hand need not be invertible, as for instance is the case for any generalized Cartan matrix of affine type. In this situation the group G admits a positive-dimensional center $Z(G)$, which leads to some complications in our study of Kac–Moody symmetric spaces. One way to resolve this issue is to consider instead of G its *adjoint quotient*

$$\mathrm{Ad}(G) := G/Z(G).$$

This group, however, has the slight disadvantage that its maximal torus is not isomorphic to a direct product of several copies of the multiplicative group $(\mathbb{R}^{\times}, \cdot)$, i.e., it is not an algebraically simply connected split torus. We thus introduce an intermediate object, that we call the *semisimple adjoint quotient* \overline{G} of G . By Proposition 3.9 below \overline{G} is the unique group which admits surjections with central kernel

$$G \rightarrow \overline{G} \rightarrow \mathrm{Ad}(G)$$

such that the kernel of the former epimorphism is a product of copies of the multiplicative group $(\mathbb{R}^{\times}, \cdot)$ and the kernel of the latter epimorphism is finite.

The construction of \overline{G} relies on some key properties of the *adjoint representation* of G and the *exponential function* of G . Both relate the complex Kac–Moody group $G_{\mathbb{C}}(\mathbf{A})$ to the (derived) complex Kac–Moody algebra \mathfrak{g} associated with \mathbf{A} , whose basic structure theory is discussed in Section AA in the appendix.

Symmetrizability of the generalized Cartan matrix as required in Convention 3.3 allows one to apply the Gabber–Kac Theorem A.3 which implies that the Lie algebra \mathfrak{g} is the direct limit of its standard subalgebras of ranks one and two. These are the Lie algebras of the standard rank one and two subgroups of $G_{\mathbb{C}}(\mathbf{A})$ and by universality the adjoint actions of these subgroups combine to an adjoint representation

$$\mathrm{Ad}_{\mathbb{C}} : G_{\mathbb{C}}(\mathbf{A}) \rightarrow \mathrm{GL}(\mathfrak{g}).$$

This restricts to a representation

$$\mathrm{Ad} : G \rightarrow \mathrm{GL}(\mathfrak{g}),$$

whose image is isomorphic to $\mathrm{Ad}(G) = G/Z(G)$.

As discussed in Section AA the Lie algebra \mathfrak{g} contains a canonical subalgebra

$$\mathfrak{h} = \sum_{i=1}^n \mathbb{C} \check{\alpha}_i$$

(see formula (A.4)), which intersects each of the standard rank one Lie algebras $\mathfrak{g}_i \cong \mathfrak{sl}(2, \mathbb{C})$ of \mathfrak{g} in the standard diagonal Cartan subalgebra $\mathfrak{h}_i := \mathbb{C} \check{\alpha}_i$ (see Theorem A.3). For each $i \in \{1, \dots, n\}$ there exists a natural exponential function

$$\exp_i : \mathfrak{h}_i < \mathfrak{g}_i \cong \mathfrak{sl}(2, \mathbb{C}) \rightarrow G_i(\mathbb{C}) \cong \mathrm{SL}(2, \mathbb{C}),$$

whose image is denoted by H_i . The groups $H_i < G$ generate the direct product

$$(\mathbb{C}^\times)^n \cong H_{\mathbb{C}} := \prod_{i=1}^n H_i < G_{\mathbb{C}}(\mathbf{A})$$

and one obtains a natural exponential function

$$\begin{aligned} \exp_{\mathbb{C}} : \mathfrak{h} = \bigoplus_{i=1}^n \mathfrak{h}_i &\rightarrow \prod_{i=1}^n H_i = H_{\mathbb{C}} \\ (X_1, \dots, X_n) &\mapsto \prod_{i=1}^n \exp_i(X_i). \end{aligned}$$

Under the standard identifications $\mathfrak{h} \cong \mathbb{C}^n$ and $H_{\mathbb{C}} \cong (\mathbb{C}^\times)^n$ this map corresponds to the usual exponential map. Recall from (A.5) on page 66 and (A.6) on page 67 that the center $\mathfrak{c} = \mathfrak{z}(\mathfrak{g})$ is contained in \mathfrak{h} and has complex dimension

$$\dim_{\mathbb{C}} \mathfrak{c} = n - \mathrm{rk}(\mathbf{A}). \quad (3.1)$$

Definition 3.5. Set $C_{\mathbb{C}} := \exp_{\mathbb{C}}(\mathfrak{c})$ and $C := C_{\mathbb{C}} \cap G = C_{\mathbb{C}} \cap G_{\mathbb{R}}(\mathbf{A})$ and define the *semisimple adjoint quotient* of G by

$$\overline{G} := G/C.$$

The *standard Cartan subgroup* of $G = G_{\mathbb{R}}(\mathbf{A})$ is defined as $T := H_{\mathbb{C}} \cap G \cong (\mathbb{R}^\times)^n$. Its image \overline{T} in \overline{G} is called the *standard Cartan subgroup* of \overline{G} .

Let now \mathfrak{a} be the real form of \mathfrak{h} defined in Notation A.4 on page 68. It is an immediate consequence of the definitions that $\exp_{\mathbb{C}}$ restricts to an injective map

$$\exp : \mathfrak{a} \rightarrow T,$$

whose image is denoted by $A := \exp(\mathfrak{a}) \cong (\mathbb{R}_{>0})^n$. Moreover, the image of A in \overline{G} is denoted by \overline{A} . The map $\exp : \mathfrak{a} \rightarrow A$ is a bijection which maps $\mathfrak{c} \cap \mathfrak{a}$ to $C \cap A$. Denoting $\overline{\mathfrak{a}} := \mathfrak{a}/(\mathfrak{c} \cap \mathfrak{a})$ as in Notation A.4, this induces a bijection

$$\exp : \overline{\mathfrak{a}} \rightarrow \overline{A}.$$

The inverse maps are denoted by $\log : A \rightarrow \mathfrak{a}$, respectively $\log : \overline{A} \rightarrow \overline{\mathfrak{a}}$. Note that, as vector spaces,

$$\mathfrak{a} \cong \mathbb{R}^n \quad \text{and} \quad \overline{\mathfrak{a}} \cong \mathbb{R}^{\text{rk}(\mathbf{A})}.$$

Remark 3.6. Before continuing we point out an error in [HKM13]. The statement of [HKM13, Lemma 7.5] is inaccurate, as becomes obvious from (3.1) above. The problem is that its proof only applies to \overline{G} (and its analogs over other fields) but not to G (or its analogs over other fields).

As a consequence, also [HKM13, Proposition 7.18] has only been established for center-free Kac–Moody groups over local fields and central quotients of \overline{G} (and its analogs over other local fields) instead of central quotients of G (or its analogs over other local fields). That is, the results from [HKM13] only enable us to control the topology on $H_{\mathbb{C}}/C_{\mathbb{C}}$ instead of the topology on $H_{\mathbb{C}}$.

However, a variation of the embedding argument as used in [HKM13, Proposition 7.10] in fact allows one to also control the topology on $H_{\mathbb{C}}$ as follows.

Proposition 3.7. *The exponential map $\exp_{\mathbb{C}} : \mathfrak{h} \rightarrow H_{\mathbb{C}}$ is a quotient map, where \mathfrak{h} is equipped with its topological vector space topology and $H_{\mathbb{C}}$ with the Kac–Peterson topology.*

Proof. It suffices to prove that the Kac–Peterson topology induces the standard topology on $H_{\mathbb{C}} \cong (\mathbb{C}^{\times})^n$. Let \mathbf{B} be an invertible generalized Cartan matrix that contains \mathbf{A} as a principal submatrix. Then $G_{\mathbb{C}}(\mathbf{A})$ admits a natural topological embedding into $G_{\mathbb{C}}(\mathbf{B})$ as a closed subgroup with respect to the Kac–Peterson topology and the subgroup $H_{\mathbb{C}}$ of $G_{\mathbb{C}}(\mathbf{A})$ embeds topologically as a closed subgroup into the corresponding subgroup $H_{\mathbb{C}}^{\mathbf{B}}$ of $G_{\mathbb{C}}(\mathbf{B})$. Since \mathbf{B} is invertible, the associated Kac–Moody algebra and group have zero-dimensional center (see formula (3.1)) and so, in fact, [HKM13, Proposition 7.18] applies to $G_{\mathbb{C}}(\mathbf{B})$; that is, $H_{\mathbb{C}}^{\mathbf{B}}$ endowed with the Kac–Peterson topology is homeomorphic to $(\mathbb{C}^{\times})^{\text{rk}(\mathbf{B})}$ endowed with its standard topology. Consequently, the closed subgroup $H_{\mathbb{C}}$ is homeomorphic to $(\mathbb{C}^{\times})^n$ endowed with its standard topology. \square

The group A carries a natural group topology induced by the Kac–Peterson topology on $H_{\mathbb{C}}$, which by Proposition 3.7 makes A homeomorphic to $(\mathbb{R}_{>0})^n$ with its standard topology. Moreover, one obtains the following immediate consequences:

Proposition 3.8. (i) *The exponential maps $\exp : \mathfrak{a} \rightarrow A$ and $\exp : \overline{\mathfrak{a}} \rightarrow \overline{A}$ are homeomorphisms, if one endows \mathfrak{a} and $\overline{\mathfrak{a}}$ with their standard vector space topologies and A and \overline{A} with the Kac–Peterson topology, respectively the induced quotient topology. In particular, the maps $\log : A \rightarrow \mathfrak{a}$ and $\log : \overline{A} \rightarrow \overline{\mathfrak{a}}$ are continuous.*

(ii) *The groups T and \overline{T} are isomorphic as topological groups to $(\mathbb{R}^{\times})^n$ and $(\mathbb{R}^{\times})^{\text{rk}(\mathbf{A})}$, respectively, and their respective identity components equal A and \overline{A} .* \square

Since $T \cong (\mathbb{R}^{\times})^n$, it contains a unique maximal finite subgroup M of order 2^n . As topological groups one has $T \cong M \times A$, where M is equipped with the discrete topology. Similarly $\overline{T} \cong \overline{M} \times \overline{A}$, where \overline{M} is the image of M in \overline{G} , which is the unique maximal finite subgroup of \overline{T} of order $2^{\text{rk}(\mathbf{A})}$.

Proposition 3.9. (i) *The kernel C of the surjection $G \rightarrow \overline{G}$ is isomorphic to $(\mathbb{R}^{\times})^{n-\text{rk}(\mathbf{A})}$ as a topological group.*

(ii) *The kernel of the map $\overline{G} \rightarrow \text{Ad}(G)$ is finite and, in fact, isomorphic to $(\mathbb{Z}/2\mathbb{Z})^k$ for some $k < n$. In particular, it is contained in \overline{M} .*

Proof. (i) follows by construction (cf. [Kac90, Proposition 1.6]). (ii) Since 1 and -1 are the only roots of unity contained in the real numbers \mathbb{R} , this follows from the proof of [HKM13, Lemma 7.5]. (Note Remark 3.6.) \square

3D The extended Weyl group

As discussed in the appendix, the generalized Cartan matrix \mathbf{A} gives rise to a quadruple $(\mathfrak{g}(\mathbf{A}), \mathfrak{h}(\mathbf{A}), \Pi, \check{\Pi})$ (see (A.1)) and a Coxeter datum (W, S, Φ, Π) (see Definition A.5). One can define W as the subgroup of $\mathrm{GL}(\mathfrak{h}(\mathbf{A}))$ generated by the set $S = \{\check{r}_{\alpha_1}, \dots, \check{r}_{\alpha_n}\}$ of reflections given by

$$\check{r}_{\alpha_i}(h) = h - \alpha_i(h)\check{\alpha}_i \quad (i = 1, \dots, n),$$

see (A.9) and also [KP85, Lemma 1.2]. The action of W on $\mathfrak{h}(\mathbf{A})$ preserves the subspace \mathfrak{a} , and descends further to the quotient $\bar{\mathfrak{a}}$ of \mathfrak{a} (see Proposition A.6).

The corresponding representations $\rho_{KM} : W \rightarrow \mathrm{GL}(\mathfrak{a})$ and $\bar{\rho}_{KM} : W \rightarrow \mathrm{GL}(\bar{\mathfrak{a}})$ are called the *Kac–Moody representation*, respectively the *reduced Kac–Moody representation* of W . Either of these representations (and their duals as well) are often also called the *geometric representation* of W . Since \mathbf{A} is assumed to be non-affine (Convention 3.3), both representations are faithful (see Corollary A.11).

Following [KP85, Corollary 2.3(b)(ii)] for every $i \in \{1, \dots, n\}$ define $\tilde{s}_{\alpha_i} \in G_i < G = G_{\mathbb{R}}(\mathbf{A})$ by

$$\tilde{s}_{\alpha_i} := \varphi_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then by [KP85, Corollary 2.3(a)] the *extended Weyl group* $\widetilde{W} := \langle \tilde{s}_{\alpha_1}, \dots, \tilde{s}_{\alpha_n} \rangle$ normalizes A and each $(\tilde{s}_{\alpha_i})^2$ is contained in A , inducing the epimorphism

$$\widetilde{W} \rightarrow W \cong N_G(T)/T, \quad \tilde{s}_{\alpha_i} \mapsto \check{r}_{\alpha_i},$$

see [KP85, Proposition 2.1]. One concludes that $\mathrm{Ad}(\tilde{s}_{\alpha_i}) \in \mathrm{GL}(\mathfrak{g})$ stabilizes \mathfrak{a} and satisfies $s_{\alpha_i} = \mathrm{Ad}(\tilde{s}_{\alpha_i})|_{\mathfrak{a}} = \rho_{KM}(W) \cong W$.

3E The twin BN pair and the twin building

Let \mathbf{A} be as in Convention 3.3, let $G = G_{\mathbb{R}}(\mathbf{A})$ as in Definition 3.4, and let $\overline{G} = G/C$ be the semisimple adjoint quotient from Definition 3.5. Both G and \overline{G} act strongly transitively on the same twin building and, hence, admit twin BN pairs (see [AB08, Theorem 8.9]).

The group G in fact admits a centered RGD system $(G, \{U_{\alpha}\}_{\alpha \in \Phi}, T)$ in the sense of Definition 3.1, called the *canonical centered RGD system* and defined as follows, cf. [R  m02, Proposition 8.4.1]: The underlying set of roots Φ equals the set of real roots of the Kac–Moody algebra $\mathfrak{g}(\mathbf{A})$, see Section AC. The group T is generated by the images of the diagonal subgroups $T_0 \subset \mathrm{SL}_2(\mathbb{R})$ under the maps φ_i from Definition 3.4 and, given a simple root α_i , one defines

$$U_{\alpha_i} := \varphi_i \left(\left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{R} \right\} \right).$$

For an arbitrary real root $\alpha \in \Phi$ one writes $\alpha = w.\alpha_i$ (see Section AC) and defines

$$U_{\alpha} := \tilde{w}U_{\alpha_i}\tilde{w}^{-1},$$

where $w = \check{r}_{\alpha_{j_1}} \cdots \check{r}_{\alpha_{j_n}} \in W$ and $\tilde{w} := \tilde{s}_{\alpha_{j_1}} \cdots \tilde{s}_{\alpha_{j_n}} \in \widetilde{W}$ as in Section 3D.

As in Section 3A denote by (B_+, B_-, N) the twin BN pair of G induced by this RGD system and by $\Delta_{\pm} := G/B_{\pm}$ the sets of chambers or the corresponding positive and negative halves of the associated twin building (cf. [AB08, Section 8.9]).

The group \overline{G} inherits an *induced centered RGD system* $(\overline{G}, \{\overline{U}_{\alpha}\}_{\alpha \in \Phi}, \overline{T})$, where $\overline{U}_{\alpha} \cong U_{\alpha}$ and \overline{T} , respectively denote the images of U_{α} and T in \overline{G} . Denote by $(\overline{B}_+, \overline{B}_-, \overline{N})$ the twin BN pair of \overline{G}

associated with the induced centered RGD system. Then, by construction, $\overline{B}_\pm = \overline{T} \ltimes \overline{U}_\pm$ where $\overline{U}_\pm := \langle \overline{U}_\alpha \mid \alpha \in \pm\Phi^+ \rangle$ as in Section 3A.

Since $C \subset T \subset B_\pm$, one has $G/B_\pm = (G/C)/(B_\pm/C) = \overline{G}/\overline{B}_\pm$. That is, the halves of the twin buildings associated with G and \overline{G} coincide. In other words, the action of G on Δ_\pm induces an action of \overline{G} on Δ_\pm . Note, furthermore, that $U_\pm \cong \overline{U}_\pm$, as by [AB08, Lemma 8.31, Corollary 8.32] both act sharply transitively on the set of chambers opposite the respective fundamental chambers in Δ_\mp .

In general, given a group with RGD system, the kernel of the action of that group on either half of the associated twin building equals the center of the group ([AB08, Proposition 8.82]). In particular, by Proposition 3.9 the action of \overline{G} on Δ_\pm has a finite kernel, whereas the action of G on Δ_\pm has an infinite kernel, if \mathbf{A} is not invertible.

We will use the following refinement of the Birkhoff decomposition. (Note that it is different from what is known as the *refined* Birkhoff decomposition in the literature). The spherical case is argued to hold in [HW93, Remark 6.5] by referring to [BT65, Theorem 5.15].

Lemma 3.10. *G and \overline{G} can be written as disjoint unions*

$$G = \bigsqcup_{n \in N_G(T)} U_+ n U_-, \quad \overline{G} = \bigsqcup_{n \in N_{\overline{G}}(\overline{T})} \overline{U}_+ n \overline{U}_-.$$

Proof. For G this is [KP85, Proposition 3.3(a), p. 181], also [Kum02, Theorem 5.2.3(g)]. Note that in the latter this is proved for a *refined* Tits system as defined in [KP83], but by [Rém02, 1.5.4], the Tits system for a group with an RGD system is indeed refined. The same argument applies to the refined Tits system for \overline{G} . \square

Definition 3.11. Given a real root $\alpha \in \Phi$ define the *rank one subgroup* as

$$G_\alpha := \langle U_\alpha, U_{-\alpha} \rangle.$$

Note that the standard rank one subgroups of G introduced in Definition 3.4 are the rank one subgroups associated with the simple roots.

By [HKM13, Proposition 7.15] (see also [KP83, Section 2E]) the subgroups B_\pm are closed in G with respect to the Kac–Peterson topology and hence Δ_\pm are Hausdorff k_ω -spaces with respect to the quotient topology by [FT77, Assertion 11, p. 116f].

The following proposition summarizes further topological properties of the various subgroups defined above:

Proposition 3.12. (i) T is closed in G and homeomorphic to $(\mathbb{R}^\times)^n$. Similarly, \overline{T} is closed in \overline{G} and homeomorphic to $(\mathbb{R}^\times)^{\text{rk}(\mathbf{A})}$.

(ii) $T \cong M \times A$ and $\overline{T} \cong \overline{M} \times \overline{A}$, where M and \overline{M} are the maximal finite subgroups and A and \overline{A} are the connected components of T and \overline{T} , respectively. Furthermore, the center of \overline{G} is contained in \overline{M} .

(iii) If the generalized Cartan matrix \mathbf{A} is two-spherical, then the set $B_+ B_-$ is open in G and multiplication induces a homeomorphism $U_+ \times T \times U_- \rightarrow B_+ B_-$; the set $\overline{B}_+ \overline{B}_-$ is open in \overline{G} and multiplication induces a homeomorphism $U_+ \times \overline{T} \times U_- \rightarrow \overline{B}_+ \overline{B}_-$.

(iv) If the generalized Cartan matrix \mathbf{A} is two-spherical, then the set $U_+ A U_-$ is open in G and multiplication $U_+ \times A \times U_- \rightarrow U_+ A U_-$ is a homeomorphism; the set $\overline{U}_+ \overline{A} \overline{U}_-$ is open in \overline{G} and multiplication $\overline{U}_+ \times \overline{A} \times \overline{U}_- \rightarrow \overline{U}_+ \overline{A} \overline{U}_-$ is a homeomorphism.

(v) Multiplication induces homeomorphisms $M \times A \times U_\pm \rightarrow B_\pm$ and $\overline{M} \times \overline{A} \times \overline{U}_\pm \rightarrow \overline{B}_\pm$.

(vi) Every rank one subgroup in G or \overline{G} is isomorphic as a topological group to $(\mathrm{P})\mathrm{SL}_2(\mathbb{R})$ with its unique connected Lie group topology, and every root subgroup is isomorphic as a topological group to $(\mathbb{R}, +)$ endowed with its standard topology.

Proof. (i) T is closed in G by [HKM13, Corollary 7.17(iii)], and so is \overline{T} in \overline{G} . The remaining statements follow from Proposition 3.8.

(ii) This follows from the discussion after Proposition 3.8 and Proposition 3.9.

(iii) follows from [HKM13, Lemma 6.1, Proposition 6.6, Proposition 7.31].

(iv) follows from (i) and (iii): Since $T = A \times M$ with M finite, A is open in T and thus $U_+ \times A \times U_- \subset U_+ \times T \times U_-$ is open. Consequently, the restriction of the open map $U_+ \times T \times U_- \rightarrow B_+ B_-$ to the open subset $U_+ \times A \times U_-$ is also open, in particular its image is open. For \overline{G} one argues similarly.

(v) follows from [HKM13, Proposition 7.27(ii)] plus assertion (ii).

(vi) is immediate by [HKM13, Corollary 7.16(iii)] and [HKM13, Corollary 7.17(ii)]. \square

Remark 3.13. It is an interesting question whether for general Cartan matrices \mathbf{A} the map $U_+ \times \overline{T} \times U_- \rightarrow \overline{B}_+ \overline{B}_-$ is open. Currently this is only known under the additional hypothesis that \mathbf{A} be two-spherical [HKM13, Proposition 7.31], but we expect that it is possible to remove this hypothesis; in fact, already Kac and Peterson had this expectation in [KP83, Section 4G]. If this expectation becomes reality, then one can remove the assumption of two-sphericity in Proposition 3.12 and consequently in a number of results below. Our suggested approach towards proving the conjecture makes use of an unfolding argument as described in [HKL15, Definition 1.10] that is very likely to allow one to embed an arbitrary symmetrizable split real Kac–Moody group G as a closed subgroup into a simply laced split real Kac–Moody group G' in such a way that the RGD systems are compatible with one another (see also [Mar15, Theorem C]). The fact that [HKM13, Proposition 7.31] applies to the ambient simply-laced Kac–Moody group G' should allow one to prove the analogous statement for the original Kac–Moody group G via (co)restrictions of the multiplication map.

Note here that (co)restrictions of open maps of course frequently fail to be open. However, since one is dealing with a bijection in this situation, one can as well establish the continuity of the inverse map, a property that behaves very well under (co)restrictions.

3F The Cartan–Chevalley involution and the twist map

Each of the standard rank one subgroups $(\mathrm{P})\mathrm{SL}_2(\mathbb{R}) \cong G_i < G$ admits a continuous involution θ_i induced by $g \mapsto (g^{-1})^T$. By [Cap09, Section 8.2] (also [KP85, Section 2]), for suitable choices of the given isomorphisms $(\mathrm{P})\mathrm{SL}_2(\mathbb{R}) \cong G_i$ these involutions θ_i extend uniquely to an involution $\theta : G \rightarrow G$, called the *Cartan–Chevalley involution* of G .

The fixed point set of θ is denoted by

$$K := G^\theta = \{k \in G \mid \theta(k) = k\}.$$

Since θ is continuous by [HKM13, Lemma 7.20], the group K is a closed subgroup of G and therefore a k_ω -topological group (cf. [FT77, p. 118]).

Proposition 3.14. *The extended Weyl group \widetilde{W} introduced in Section 3D is contained in K .*

Proof. For $i \in \{1, \dots, n\}$

$$\tilde{s}_{\alpha_i} := \varphi_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in G_i^{\theta_i} < K,$$

and these generate \widetilde{W} . \square

Lemma 3.15. *The Cartan–Chevalley involution fixes T and maps U_+ to U_- . In particular, $\theta(B_+) = B_-$.*

Proof. This follows from the observation that on each of the rank one subgroups, θ preserves the diagonal subgroup and interchanges the groups $U_{\alpha_i} = \varphi_i \left(\left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{R} \right\} \right)$ and $U_{-\alpha_i} = \varphi_i \left(\left\{ \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \mid t \in \mathbb{R} \right\} \right)$. \square

Proposition 3.16. *The Cartan–Chevalley involution preserves C and hence induces a continuous involution $\bar{\theta}$ of \bar{G} , which fixes \bar{T} and maps \bar{B}_+ to \bar{B}_- .*

Proof. Let $d\theta : \mathfrak{g} \rightarrow \mathfrak{g}$ be the involution of \mathfrak{g} which on the rank one subalgebras $\mathfrak{g}_i \cong \mathfrak{sl}_2(\mathbb{C})$ is given by $X \mapsto -X^*$. This satisfies $d\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$ (cf. [Kac90, p. 7]) for every root α , and in particular preserves $\ker(\alpha_i) \subset \mathfrak{h}$ for every $i \in \{1, \dots, n\}$. It thus follows from the definition of \mathfrak{c} in (A.5) on page 66 that the latter is $d\theta$ -invariant. Since $\exp_{\mathbb{C}}$ intertwines $d\theta$ and θ (the latter considered as an automorphism of $G_{\mathbb{C}}(\mathbf{A})$), it follows that θ preserves C . The other statements now follow from Lemma 3.15. \square

Note that the image of K in \bar{G} is equal to $\bar{K} := \bar{G}^{\bar{\theta}}$, as both groups are generated by the panel stabilizers $\bar{G}_i^{\bar{\theta}} \bar{T}^{\bar{\theta}}$, $1 \leq i \leq n$.

Let us recall and adjust to our setting some of the notions introduced in [Ric82, Section 2]; see also [HW93, Section 6], [KW92, Section 5].

Definition 3.17. Let $G = G_{\mathbb{R}}(\mathbf{A})$ be the simply connected split real Kac–Moody group of type \mathbf{A} , let θ be its Cartan–Chevalley involution, let \bar{G} be the semisimple adjoint quotient of G , and let $\bar{\theta}$ be the involution of \bar{G} induced by θ .

(i) The maps

$$G \times G \rightarrow G, \quad (g, x) \mapsto g * x := gx\theta(g)^{-1} \quad \text{and} \quad \bar{G} \times \bar{G} \rightarrow \bar{G}, \quad (g, x) \mapsto g * x := gx\bar{\theta}(g)^{-1}$$

are called the *twisted conjugation maps* of G and \bar{G} , respectively.

(ii) The *twist maps* of G , respectively \bar{G} are the continuous map

$$\tau : G \rightarrow G, \quad g \mapsto g * e = g\theta(g)^{-1} \quad \text{and} \quad \bar{\tau} : \bar{G} \rightarrow \bar{G}, \quad g \mapsto g * e = g\bar{\theta}(g)^{-1}.$$

Note that twisted conjugation defines a left-action of G on itself, since

$$g * (h * x) = g * (hx\theta(h)^{-1}) = ghx\theta(h)^{-1}\theta(g)^{-1} = (gh)x\theta(gh)^{-1} = (gh) * x,$$

while τ is an orbit map of this group action; a similar statement holds for \bar{G} . The following lemma summarizes various basic properties of the twist map.

Lemma 3.18. (i) For $g \in \tau(G)$ one has $\theta(g) = g^{-1}$ and $\tau(g) = g^2$.

(ii) For $g, h \in G$ one has $\tau(gh) = g * \tau(h)$.

(iii) $\tau^{-1}(e) = K$.

(iv) For $g, h \in G$ one has $gK = hK \Leftrightarrow \tau(g) = \tau(h) \Leftrightarrow \tau(h^{-1}g) = e$.

(v) For every $S \subseteq G$ one has $\tau^{-1}(\tau(S)) = SK$.

(vi) τ factors through G/K , yielding a surjective map

$$\hat{\tau} : G/K \rightarrow \tau(G), \quad gK \mapsto \tau(g).$$

Analogous statements hold for \bar{G} instead of G .

Remark 3.19. In fact, Definition 3.17 makes sense for an arbitrary group G with involution $\theta \in \text{Aut}(G)$, and Lemma 3.18 remains valid in this generality for $K := G^\theta$. In this broader context, one sees that the twist map from Definition 3.17 can be considered as a non-Galois version of the famous Lang map from [Lan56, Section 2].

Furthermore, even in the case of real Kac–Moody groups there exist involutions θ different from the Cartan–Chevalley involution that lead to symmetric spaces G/G^θ worthwhile of further study; we refer to [KW92] and [GHM11] for a discussion of abstract involutions of Kac–Moody algebras and Kac–Moody groups that might provide a starting point for studying these more general Kac–Moody symmetric spaces.

Proof of Lemma 3.18. (i) For $g = h\theta(h)^{-1} \in \tau(G)$ one computes $\theta(g) = \theta(h)\theta(\theta(h)^{-1}) = \theta(h)h^{-1} = g^{-1}$ and $\tau(g) = h\theta(h)^{-1}\theta(h\theta(h)^{-1})^{-1} = (h\theta(h)^{-1})^2 = g^2$.

(ii) One has $\tau(gh) = gh * e = g * (h * e) = g * \tau(h)$.

(iii) For $g \in G$, one has $\tau(g) = e \Leftrightarrow g\theta(g)^{-1} = e \Leftrightarrow g = \theta(g) \Leftrightarrow g \in K$.

(iv) One computes

$$\begin{aligned} gK = hK &\Leftrightarrow \exists k \in K : g = hk \implies \tau(g) = \tau(hk) = \tau(h) \implies g\theta(g)^{-1} = h\theta(h)^{-1} \\ &\implies h^{-1}g = \theta(h)^{-1}\theta(g) = \theta(h^{-1}g) \implies h^{-1}g \in K \implies gK = hK. \end{aligned}$$

Moreover, by (iii), one has $h^{-1}g \in K \Leftrightarrow \tau(h^{-1}g) = e$.

(v) Let $B := \tau(S)$. Then $x \in \tau^{-1}(B) \Leftrightarrow \tau(x) \in B \Leftrightarrow \exists s \in S : \tau(x) = \tau(s) \Leftrightarrow \exists s \in S : xK = sK \Leftrightarrow x \in SK$.

(vi) follows from (v). □

Lemma 3.20. (i) $\tau(t) = t^2$ for all $t \in T$.

(ii) $A = \tau(T) = \tau(A)$.

(iii) $B_\pm \cap K = T \cap K = M$ and $A \cap K = \{e\}$.

Proof. The key observation is that T is the direct product of the diagonal subgroups $T_i \cong \mathbb{R}^\times$ in G_i , and on each of the T_i the involution θ acts by inversion. In particular, $\tau(t) = t\theta(t)^{-1} = t^2$ for all $t \in T_i$, whence (i) follows. Since the set of squares in \mathbb{R}^\times is given by $\mathbb{R}^{>0}$, and every element in $\mathbb{R}^{>0}$ has a positive square root, (ii) follows from (i). Concerning (iii), observe first that, if $g \in B_\pm \cap K$, then $\theta(g) = g$. Since $\theta(B_\pm) = B_\mp$, this implies $g \in B_+ \cap B_- = T$, so $B_\pm \cap K = T \cap K$. Now let $t \in T$ and write $t = ma$ with $a \in A \cong (\mathbb{R}^\times)^n$, $m \in M = (\mathbb{Z}/2\mathbb{Z})^n$ (see Definition 3.5). Then $\tau(t) = m^2a^2 = a^2$, and thus $\tau(t) = e$ if and only if $a = e$, as A is torsion-free. Hence $T \cap K = M$ and $A \cap K = \{e\}$ by Lemma 3.18(iii). □

Lemma 3.21. $N_G(T) = A \rtimes N_K(T)$.

Proof. Observe that, since K acts transitively on Δ_\pm , the group $N_K(T)$ acts transitively on the θ -stable apartment fixed by T . Therefore $N_G(T)/T \cong N_K(T)/T \cap N_K(T) \cong N_K(T)T/T$ and so $N_G(T) = N_K(T)T$. Furthermore, by Lemma 3.20 on one hand one has $\tau(t) = t^2$ for all $t \in T$ and on the other hand $M = K \cap T \subset N_K(T)$. Hence $N_G(T) = N_K(T)T = N_K(T)MA = N_K(T)A$. Since $A \subset T$ is normalized by $N_K(T)$ and $A \cap N_K(T) \subset A \cap K = \{e\}$ (see Lemma 3.20), one arrives at $N_G(T) = A \rtimes N_K(T)$. □

The following technical observation depends heavily on the language of twin buildings. Refer to [AB08, Sections 5.8 and 6.3] and [Hor17] for the necessary background information.

Lemma 3.22 ([Hor17, Lemma 4.2]). *Suppose $g \in G$ is symmetric, i.e., $\theta(g) = g^{-1}$. Then the following assertions concerning the action of g on the twin building of G are equivalent:*

- (i) g fixes a θ -stable apartment chamberwise.
- (ii) g fixes an apartment chamberwise.
- (iii) g stabilizes a chamber.
- (iv) g has a bounded (with respect to the gallery metric) orbit.
- (v) g stabilizes a spherical residue. □

3G The topological Iwasawa decomposition

The goal of this subsection is to prove the following decomposition results for G and \overline{G} .

Theorem 3.23 (Topological Iwasawa decomposition). *Let $G = G_{\mathbb{R}}(\mathbf{A})$ be the simply connected split real Kac–Moody group of type \mathbf{A} and let \overline{G} be its semisimple adjoint quotient.*

- (i) $K \cap B_{\pm} = M$ and $\overline{K} \cap \overline{B}_{\pm} = \overline{M}$. In particular, the center of \overline{G} is contained in \overline{K} .
- (ii) Multiplication induces continuous bijections $m_1 : U_{\pm} \times A \times K \rightarrow G$, $m_2 : K \times A \times U_{\pm} \rightarrow G$ and homeomorphisms $\overline{m}_1 : \overline{U}_{\pm} \times \overline{A} \times \overline{K} \rightarrow \overline{G}$ and $\overline{m}_2 : \overline{K} \times \overline{A} \times \overline{U}_{\pm} \rightarrow \overline{G}$.
- (iii) The action of K on both halves of the twin building factors through \overline{K} , which acts transitively on both halves of the twin building. Moreover, $\Delta_{\pm} \cong \overline{K}/\overline{M}$, where $\overline{K}/\overline{M}$ carries the quotient topology.

Proof of Theorem 3.23, discrete version. First establish the results concerning G . (i) follows from Lemma 3.20. Concerning (iii), recall from [DMGH09] that $G = KB_{\pm}$. In particular, K acts transitively on Δ_{\pm} .

Now consider the map m_1 from (ii). Since $B_{\pm} = MAU_{\pm}$ and $G = KB_{\pm}$, one has $G = KMAU_{\pm} = KAU_{\pm}$, i.e. m_1 is surjective. Injectivity of m_1 follows from $B_{\pm} \cap K = M$, so that m_1 is a bijection. Since inversion intertwines m_1 and m_2 , it follows that also m_2 is bijective, establishing the discrete part of Theorem 3.23 for G .

Concerning \overline{G} , since the action of K on Δ_{\pm} factors through \overline{K} , the latter acts transitively on Δ_{\pm} , i.e. $\overline{G} = \overline{K}\overline{B}_{\pm}$. The fact $K \cap B_{\pm} = M < T$ implies $\overline{K} \cap \overline{B}_{\pm} = \overline{M} < \overline{T}$. In particular, $\Delta_{\pm} \cong \overline{K}/\overline{M}$ as sets. This in turn implies bijectivity of \overline{m}_1 by the same argument used to show bijectivity of m_1 and, thus, also of \overline{m}_2 . The statement about the center of \overline{G} follows from Proposition 3.12(ii). □

Lemma 3.24. $N_G(T) \cap \tau(G) = A$.

Proof. Suppose $g \in N_G(T) \cap \tau(G)$. On the one hand, since $N_G(T) = A \rtimes N_K(T)$ by Lemma 3.21, there exist unique $t \in A$ and $k \in K$ such that $g = tk$. On the other hand $g \in \tau(G)$, so by the (discrete version of the) Iwasawa decomposition (see Theorem 3.23) there exist $u \in U_+$, $t' \in A$, $k' \in K$ such that $g = \tau(ut'k') = \tau(ut')$, whence

$$g = \tau(ut') = ut'^2\theta(u)^{-1} \in U_+t'^2U_-.$$

Since $g \in N_G(T)$, the Birkhoff decomposition (see Lemma 3.10) yields $t'^2 = g = tk$. Now $t' \in A$ implies $t'^2 \in A$ and thus $k = t^{-1}t'^2 \in A$. One concludes with Lemma 3.20 that $k = e$, hence $g = t \in A$ as claimed. □

Before turning to the topological version of the theorem, recall some basic facts about k_{ω} -spaces, cf. [FT77].

Definition 3.25. A Hausdorff topological space X is called a k_ω -space, if it is the direct limit of an increasing family of compact subspaces $(X_n)_{n \in \mathbb{N}}$, i.e., if $X = \bigcup_n X_n$ and a subset Y of X is open in X if and only if each intersection $Y \cap X_n$ is open in X_n ; the increasing family $(X_n)_{n \in \mathbb{N}}$ is called a k_ω -sequence for X and the pair $(X, (X_n))$ is called a k_ω -pair.

Lemma 3.26. Let $(X, (X_n))$ and $(Y, (Y_n))$ be k_ω -pairs and let $f : X \rightarrow Y$ be a continuous bijection such that

$$\forall n \in \mathbb{N} \quad \exists m \in \mathbb{N} : f(X_m) \supset Y_n.$$

Then $(f(X_n))$ is a k_ω -sequence for Y and f is a homeomorphism.

Proof. Since Y is Hausdorff, the sets $f(X_n)$ are compact. Hence by [FT77, Assertion 7, p. 114] for every $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ such that $f(X_n) \subset Y_k$. The hypothesis therefore implies that the sequences $(f(X_n))$ and (Y_n) define the same limit topology on Y , i.e. $(f(X_n))$ is a k_ω -sequence for Y . Now for each n the map $f : X_n \rightarrow f(X_n)$ is a homeomorphism, and hence f yields a homeomorphism

$$f : X = \lim_{\rightarrow} X_n \rightarrow \lim_{\rightarrow} f(X_n) = Y. \quad \square$$

Lemma 3.27. Let $(X, (X_n))$ be a k_ω -pair and let $\pi : \tilde{X} \rightarrow X$ is a finite-sheeted covering. Then $(\tilde{X}, \pi^{-1}(X_n))$ is a k_ω -pair.

Proof. Since π is a finite-sheeted covering, it is proper and, hence, $\tilde{X}_n := \pi^{-1}(X_n)$ is compact for every $n \in \mathbb{N}$. Since π is open, a subset $U \subset \tilde{X}$ is open if and only if $V := \pi(U)$ is open, and the same holds for the restrictions $\pi|_{\tilde{X}_n} : \tilde{X}_n \rightarrow X_n$. One thus obtains the chain of equivalences

$$\begin{aligned} U \subset \tilde{X} \text{ open} &\Leftrightarrow \pi(U) \cap X_n \text{ open in } X_n \text{ for every } n \in \mathbb{N} \\ &\Leftrightarrow \pi(U \cap \tilde{X}_n) \text{ open in } X_n \text{ for every } n \in \mathbb{N} \\ &\Leftrightarrow U \cap \tilde{X}_n \text{ open in } \tilde{X}_n \text{ for every } n \in \mathbb{N}. \end{aligned}$$

This shows that $(\tilde{X}, (\tilde{X}_n))$ is a k_ω -pair. \square

Let $\Delta_\pm = \overline{G}/\overline{B}_\pm$ denote one half of the twin building Δ . Recall from Proposition 3.12(v) that \overline{B}_+ has the decomposition $\overline{B}_+ = \overline{M}\overline{A}\overline{U}_+$, where $\overline{M} = \overline{K} \cap \overline{B}_+$ is a finite group. Denote by $\tilde{\Delta}_\pm$ the quotient $\overline{G}/\overline{A}\overline{U}_\pm$. Then the canonical projections

$$\pi_\pm : \tilde{\Delta}_\pm \rightarrow \Delta_\pm \tag{3.2}$$

are finite-sheeted covering maps with fiber \overline{M} .

Proposition 3.28. The maps

$$\iota_\pm : \overline{K} \rightarrow \tilde{\Delta}_\pm, \quad k \mapsto k\overline{A}\overline{U}_\pm$$

are homeomorphisms.

Proof. It follows from the abstract Iwasawa decomposition that ι_\pm are continuous bijections. Let

$$\overline{G}_k^\pm := \bigcup_{w \in W, l(w) \leq k} \overline{B}_\pm w \overline{B}_\pm,$$

denote by $\tilde{\Delta}_{k,\pm}$ and $\Delta_{k,\pm}$ the respective image of \overline{G}_k^\pm in $\tilde{\Delta}_\pm$ and Δ_\pm , and let $\overline{K}_k^\pm := \overline{K} \cap \overline{G}_k^\pm$. Then by [HKM13, Corollary 7.11] and the observation that direct limits commute with quotients one has

$$\overline{G} = \lim_{\rightarrow} \overline{G}_k^\pm, \tag{3.3}$$

and, thus,

$$\overline{K} = \lim_{\rightarrow} \overline{K}_k^+.$$

The subsets $\overline{K}_k^\pm \subset \overline{K}$ are compact: Indeed, \overline{K}_k^\pm equals the finite union of products of the form $K_{\alpha_1} \cdots K_{\alpha_k}$, where each $K_{\alpha_i} \cong \mathrm{SO}_2(\mathbb{R})$ is compact. Since multiplication is continuous and \overline{K} is Hausdorff, this implies that \overline{K}_k^\pm are compact, and hence $(\overline{K}, (\overline{K}_k^\pm))$ is a k_ω -pair.

By the discrete version of Theorem 3.23, the group \overline{K} acts transitively on $\tilde{\Delta}_\pm$ and one has $\iota_\pm(\overline{K}_k^\pm) = \tilde{\Delta}_{k,\pm}$. In particular, the spaces $\tilde{\Delta}_{k,\pm}$ are compact. Therefore $(\tilde{\Delta}_\pm, (\tilde{\Delta}_{k,\pm}))$ is a k_ω -pair and the proposition follows from Lemma 3.26. \square

Proof of Theorem 3.23. Assertion (i) has already been proved for the discrete version of the theorem. Concerning (iii), the finite-sheeted coverings $\pi_\pm : \tilde{\Delta}_\pm \rightarrow \Delta_\pm$ from (3.2) are continuous and open. By Proposition 3.28 this implies that the orbit maps $\overline{K} \mapsto \Delta_\pm$ are continuous and open, hence $\Delta_\pm \cong \overline{K}/\overline{M}$ as topological spaces as claimed.

In order to prove (ii), it is clear that the maps under consideration are continuous, since they are induced by the group multiplication. It thus remains to show that \overline{m}_2 , and hence \overline{m}_1 , are open. Given $g \in \overline{G}$, define $k(g) := \iota_\pm^{-1}(g\overline{A}\overline{U}_\pm)$, where ι_\pm is as in Proposition 3.28 and let $b(g) := k(g)^{-1}g$. Since ι_\pm is open and \overline{G} is a topological group, one obtains a continuous map

$$i_\pm : \overline{G} \rightarrow \overline{K} \times \overline{A}\overline{U}_\pm, \quad g \mapsto (k(g), b(g))$$

such that $g = k(g)b(g)$. This map is inverse to the multiplication map $m : \overline{K} \times \overline{A}\overline{U}_\pm \rightarrow \overline{G}$, showing that m is a homeomorphism. It remains to see that the multiplication map $\overline{A} \times \overline{U}_\pm \rightarrow \overline{A}\overline{U}_\pm$ is open; this however follows from [HKM13, Proposition 7.27]. This finishes the proof of Theorem 3.23. \square

3H The image of the twist map

It remains to understand the images of the twist maps inside their ambient groups.

Proposition 3.29. $K \cap \tau(G) = \{e\}$ and $\overline{K} \cap \overline{\tau}(\overline{G}) = \{e\}$.

Proof. Suppose $g \in K \cap \tau(G)$. Then $g = \theta(g) = g^{-1}$ by Lemma 3.18(i), so g has order 1 or 2. Hence its orbits are bounded, and so by Lemma 3.22 it stabilizes a chamber c in the twin building of G . But then also $\theta(c) = \theta(g.c) = g.\theta(c)$, so g fixes chamberwise the (unique) θ -stable twin apartment containing the two opposite chambers c and $\theta(c)$ and, thus, is contained in the corresponding θ -split torus T' of G (where θ -split means that θ leaves T' invariant and acts via inversion on T'). Since $K = G^\theta$ acts transitively on each half of the twin building, there exists $k \in K$ with ${}^kT' = T$. Thus $k * g = {}^k g \in T \cap \tau(G) \cap K$ and, by Lemma 3.24, in fact ${}^k g \in A$. But then ${}^k g \in A \cap K = \{e\}$, hence $g = e$, i.e., $K \cap \tau(G) = \{e\}$.

Similarly, one proves $\overline{K} \cap \overline{\tau}(\overline{G}) = \{e\}$. \square

Proposition 3.30. The group G (respectively, \overline{G}) is generated by its subset $\tau(G)$ (respectively, $\overline{\tau}(\overline{G})$).

Proof. The map τ preserves each of the fundamental rank one subgroups $G_i \cong (\mathrm{P})\mathrm{SL}_2(\mathbb{R})$. A simple computation in $(\mathrm{P})\mathrm{SL}_2(\mathbb{R})$ shows that $\tau(G_i)$ generates G_i (the matrix group $\mathrm{SL}_2(\mathbb{R})$ is generated by the set of positive definite symmetric matrices). Thus $\langle \tau(G) \rangle \leq G$ contains each of the fundamental rank one subgroups, whence coincides with G . The proof for \overline{G} is the same. \square

Proposition 3.31. The following assertions hold.

(i) $\tau(G) = \tau(U_+A) = U_+ * A$ and $\overline{\tau}(\overline{G}) = \overline{\tau}(\overline{U}_+\overline{A}) = \overline{U}_+ * \overline{A}$.

(ii) $\tau(G) \subset U_+AU_- \subset G$ and $\overline{\tau}(\overline{G}) \subset \overline{U}_+\overline{A}\overline{U}_- \subset \overline{G}$; more precisely,

$$\tau(G) = \{u_+au_- \in U_+AU_- \mid u_- = \theta(u_+)^{-1}\}, \quad \overline{\tau}(\overline{G}) = \{u_+\overline{a}u_- \in \overline{U}_+\overline{A}\overline{U}_- \mid u_- = \theta(u_+)^{-1}\}.$$

- (iii) Every $g \in \tau(G)$ (respectively, $g \in \tau(\overline{G})$) can be written as $g = \tau(u_1 \cdots u_m t)$ with $t \in A$ (respectively, $t \in \overline{A}$) and $u_i \in U_{\beta_i}$ (respectively, $u_i \in \overline{U}_{\beta_i}$) for some $\beta_i \in \Phi^+$.
- (iv) If the generalized Cartan matrix \mathbf{A} is two-spherical, then every $g \in \tau(G)$ (respectively, $g \in \tau(\overline{G})$) can be written as $g = \tau(u_1 \cdots u_m t)$ with $t \in A$ (respectively, $t \in \overline{A}$) and $u_i \in U_{\beta_i}$ (respectively, $u_i \in \overline{U}_{\beta_i}$) for some $\beta_i \in \Pi$.

Proof. By the Iwasawa decomposition (see Theorem 3.23), every $g \in G$ can be written as $g = u h k$ with $u \in U_+$, $h \in A$ and $k \in K$. Then $x := \tau(g) = \tau(u h) = u * \tau(h)$ by Lemma 3.18. Now $\tau(A) = A$ by Lemma 3.20, and hence $\tau(G) = U_+ * A$. Assertion (i) follows.

If $u_+ \in U_+$ and $h \in A$, then $\tau(u_+ h) = u_+ * \tau(h) = u_+ h^2 \theta(u_+)^{-1}$ by Lemma 3.20. Moreover, $h^2 \in A$ and $\theta(u_+)^{-1} \in U_-$ by Lemma 3.15. Thus (ii) follows from (i) and the fact that every element of A has a square root in A .

Finally, since A normalizes U_+ , it follows from (i) that $\tau(G) = \tau(U_+ A) = \tau(A U_+)$. Then (iii) and (iv) follow from the fact that U_+ is generated by the $(U_\alpha)_{\alpha \in \Phi^+}$ (see [AB08, Theorem 8.84]) and even by the $(U_\alpha)_{\alpha \in \Pi}$ in the two-spherical case (see [DM07, Corollary 1.2] and note from its proof that two-sphericity suffices for the generation result, only the validity of the given presentation requires three-sphericity).

The proofs for \overline{G} are similar. \square

Assume that the generalized Cartan matrix \mathbf{A} is two-spherical, let $m : U_+ \times A \times U_- \rightarrow U_+ A U_-$ be the homeomorphism from Proposition 3.12(iv) and define $h : \tau(G) \rightarrow U_+ \times A$ to be the composition

$$\tau(G) \hookrightarrow U_+ A U_- \xrightarrow{m^{-1}} U_+ \times A \times U_- \rightarrow U_+ \times A,$$

where the first map is the inclusion and the last map is the canonical projection that forgets the last component.

Corollary 3.32. *If the generalized Cartan matrix \mathbf{A} is two-spherical, then the map $h : \tau(G) \rightarrow U_+ \times A$ is a homeomorphism whose inverse is given by*

$$\begin{aligned} h^{-1} : U_+ \times A &\rightarrow \tau(G) \\ (u_+, h) &\mapsto u_+ h \theta(u_+)^{-1}. \end{aligned}$$

Proof. h is continuous, since m is open. Since θ is continuous and G is a topological group, the map $U_+ \times A \rightarrow \tau(G)$, $(u_+, h) \mapsto u_+ h \theta(u_+)^{-1}$ is continuous. By Proposition 3.31 it is onto $\tau(G)$, and it is clearly inverse to h . \square

The same argument also shows that there is a homeomorphism

$$\tau(\overline{G}) \rightarrow \overline{U}_+ \times \overline{A},$$

given by the same formula.

4 Models for Kac–Moody symmetric spaces

4A Topological symmetric spaces from involutions

Let G be an arbitrary topological group, let $\theta \in \text{Aut}(G)$ be a continuous involution and let $K = G^\theta$. In this generality one can introduce a twist map

$$\begin{aligned} \tau : G &\rightarrow G \\ g &\mapsto g \theta(g)^{-1} \end{aligned}$$

as in Definition 3.17, which will satisfy the properties described in Lemma 3.18. Since θ is continuous, K is a closed subgroup of G , and thus G/K is a Hausdorff topological space with respect to the quotient topology. Using the involution θ and the associated twist map τ one defines a multiplication map

$$\begin{aligned} \mu : G/K \times G/K &\rightarrow G/K \\ (gK, hK) &\mapsto \tau(g)\theta(h)K. \end{aligned} \quad (4.1)$$

Note that μ is continuous, since τ , θ and the group multiplication are.

Proposition 4.1. *If*

$$K \cap \tau(G) = \{e\}, \quad (4.2)$$

then the pair $(G/K, \mu)$ is a topological symmetric space and the natural action

$$\begin{aligned} G &\rightarrow \text{Sym}(G/K) \\ g &\mapsto (aK \mapsto gaK) \end{aligned}$$

is by automorphisms.

Proof. For $a, b, c \in G$ one computes:

$$(RS1) \quad \mu(aK, aK) = \tau(a)\theta(a)K = aK,$$

$$(RS2) \quad \mu(aK, \mu(aK, bK)) = \mu(aK, \tau(a)\theta(b)K) = \tau(a)\theta(\tau(a)\theta(b))K = bK,$$

$$\begin{aligned} (RS3) \quad \mu(aK, \mu(bK, cK)) &= \mu(aK, \tau(b)\theta(c)K) = \tau(a)\theta(\tau(b)\theta(c))K \\ &= \tau(a)\theta(b)b^{-1}\theta(\theta(c))K \\ &= \tau(a)\theta(b)b^{-1}\tau(a)\theta(\tau(a)\theta(c))K \\ &= \tau(\tau(a)\theta(b))\theta(\tau(a)\theta(c))K \\ &= \mu(\tau(a)\theta(b)K, \tau(a)\theta(c)K) = \mu(\mu(aK, bK), \mu(aK, cK)), \end{aligned}$$

$$\begin{aligned} (RS4) \quad \mu(aK, bK) = bK &\Leftrightarrow \tau(a)\theta(b)K = bK \Leftrightarrow b^{-1}a\theta(a)^{-1}\theta(b) = \tau(b^{-1}a) \in K \\ &\stackrel{(4.2)}{\Leftrightarrow} \tau(b^{-1}a) = e \Leftrightarrow \tau(a) = \tau(b) \Leftrightarrow aK = bK. \end{aligned}$$

Since μ is continuous, this establishes that $(G/K, \mu)$ is a topological symmetric space. The second statement follows from the fact that for $a, b, g \in G$ one has

$$\mu(gaK, gbK) = \tau(ga)\theta(gb)K = g\tau(a)\theta(g)^{-1}\theta(g)\theta(b)K = g\mu(aK, bK). \quad \square$$

4B Reduced and unreduced Kac-Moody symmetric spaces

We are now ready to associate symmetric spaces with a large class of Kac-Moody groups. We choose to work in the following general setting.

Convention 4.2. *The matrix $\mathbf{A} \in M_n(\mathbb{Z})$ denotes a generalized Cartan matrix of size $n \times n$ and rank $l \leq n$, subject to the restrictions given in Convention 3.3. That is, \mathbf{A} is assumed to be non-spherical, non-affine, irreducible, and symmetrizable.*

The group $G := G_{\mathbb{R}}(\mathbf{A})$ denotes the associated simply connected centered split real Kac-Moody group, and \overline{G} denotes its semisimple adjoint quotient. θ and $\overline{\theta}$ denote the Cartan-Chevalley involutions on G , respectively \overline{G} , and K and \overline{K} denote their respective fixed point groups.

Recall from Proposition 3.29 that $K \cap \tau(G) = \{e\}$ and $\overline{K} \cap \overline{\tau}(\overline{G}) = \{e\}$. It thus follows from Proposition 4.1 that both G/K and $\overline{G}/\overline{K}$ carry the structure of a topological symmetric space given by $(gK, hK) \mapsto \mu(gK, hK) = \tau(g)\theta(h)K$.

Definition 4.3. (i) $(G/K, \mu)$ is called the *unreduced Kac-Moody symmetric space* associated with \mathbf{A} .

(ii) $(\overline{G}/\overline{K}, \overline{\mu})$ is called the *reduced Kac-Moody symmetric space* associated with \mathbf{A} .

If \mathbf{A} is invertible, then by Proposition 3.9(i) both versions of the Kac-Moody symmetric space coincide; in this case they are referred to as *the* Kac-Moody symmetric space associated with \mathbf{A} . In general, however, these two spaces behave quite differently. Note that $\overline{G}/\overline{K} = \text{Ad}(G)/\text{Ad}(K)$, since the center of \overline{G} is contained in \overline{K} by Theorem 3.23(i), i.e., the three different groups G , \overline{G} , $\text{Ad}(G)$ do not lead to a third version of a Kac-Moody symmetric space.

A first observation is that the unreduced Kac-Moody symmetric space $(G/K, \mu)$ fibers over the reduced Kac-Moody symmetric space with fiber \mathbb{E}^{n-l} .

Proposition 4.4. (i) The canonical projection $\pi_{\mathbf{A}} : G/K \rightarrow \overline{G}/\overline{K}$ is a morphism of topological reflection spaces.

(ii) The fiber $\pi_{\mathbf{A}}^{-1}(e\overline{K})$ is isomorphic to \mathbb{E}^{n-l} as a topological reflection space.

Proof. (i) Denote the projection $G \rightarrow \overline{G}$ by $g \mapsto [g]$ so that $\pi_{\mathbf{A}}$ is given by $\pi_{\mathbf{A}}(gK) = [g]\overline{K}$. Then for all $g, h \in G$ one has

$$\pi_{\mathbf{A}}(gK) \cdot \pi_{\mathbf{A}}(hK) = [g]\overline{K} \cdot [h]\overline{K} \stackrel{(4.1)}{=} \tau([g])\overline{\theta}([h])\overline{K} = [\tau(g)\theta(h)]\overline{K} = \pi_{\mathbf{A}}(\tau(g)\theta(h)K) = \pi_{\mathbf{A}}(gK \cdot hK).$$

(ii) By definition, $\pi_{\mathbf{A}}^{-1}(e\overline{K}) = CK/K \cong C/(C \cap K) \cong (\mathbb{R}_{>0})^{n-\text{rk}(\mathbf{A})}$, where the second isomorphism follows from Proposition 3.9 and Lemma 3.20(iii). One can parametrize this fiber via

$$\varphi_o : \mathfrak{c} \cap \mathfrak{a} \rightarrow C/(C \cap K), \quad X \mapsto \exp(X)(C \cap K).$$

By endowing the vector space $\mathfrak{c} \cap \mathfrak{a}$ with its Euclidean reflection space structure this map becomes an isomorphism of reflection spaces. Indeed, if $X, Y \in \mathfrak{c} \cap \mathfrak{a}$, then

$$\begin{aligned} \varphi_o(X) \cdot \varphi_o(Y) &= \exp(X)(C \cap K) \exp(Y)(C \cap K) = \tau(\exp(X))\theta(\exp(Y))(C \cap K) \\ &= \exp(X)^2 \exp(Y)^{-1}(C \cap K) = \exp(2X - Y)(C \cap K) \\ &= \varphi_o(X \cdot Y). \end{aligned}$$

Thus the parametrization is an abstract isomorphism of reflection spaces and, in fact, a topological isomorphism by Proposition 3.8. \square

Lemma 4.5. The kernel of the action of G on G/K equals the centralizer $C_K(G)$ of G in K and the kernel of the action of \overline{G} on $\overline{G}/\overline{K}$ equals the center $Z(\overline{G})$ of \overline{G} .

Proof. Since $C_K(G) < K$, it acts trivially on G/K : for all $g \in C_K(G)$, $a \in G$ one has $gaK = agK = aK$. On the other hand, if $g \in G$ acts trivially on G/K , then for all $h \in G$ one has $ghK = hK$. In particular $gK = K$, i.e. $g \in K$ and, thus, $\theta(g) = g$. Lemma 3.18 implies

$$\begin{aligned} hgh^{-1} \in K &\Rightarrow \tau(h^{-1}gh) = e \quad \Rightarrow h^{-1} * \tau(gh) = e \\ &\Rightarrow g * \tau(h) = h * e \quad \Rightarrow g * \tau(h) = \tau(h) \\ &\Rightarrow g\tau(h)g^{-1} = \tau(h). \end{aligned}$$

Thus g centralizes $\tau(G)$. Since $\tau(G)$ generates G (see Proposition 3.30), the element g therefore centralizes G , i.e., $g \in C_K(G)$. The same argument shows that $g \in \overline{G}$ acts trivially on $\overline{G}/\overline{K}$ if and only if $g \in C_{\overline{K}}(\overline{G}) = Z(\overline{G})$ (cf. Theorem 3.23(i)). \square

Definition 4.6. Define

$$G_{\text{eff}} := G/C_K(G).$$

By Lemma 4.5 the group G_{eff} then acts effectively (i.e., faithfully) on G/K . Similarly, $\text{Ad}(G) = \overline{G}/Z(\overline{G})$ acts effectively on $\overline{G}/\overline{K}$.

Remark 4.7. By the topological Iwasawa decomposition Theorem 3.23 there exists a homeomorphism

$$U_+ \times \overline{A} \rightarrow \overline{G}/\overline{K}, \quad (u, a) \mapsto ua\overline{K}.$$

This allows one to define the structure of a topological symmetric space on $U_+ \times \overline{A}$ by transporting the multiplication map via this homeomorphism. Unfortunately, at the moment we do not know of any good way of describing this induced multiplication map intrinsically, nor do we have an intrinsic description for the induced G -action on $U_+ \times \overline{A}$.

The key problem is to derive a formula of how to decompose a product $(k_1 a_1 u_1)(k_2 a_2 u_2)$ with respect to $\overline{K} \times \overline{A} \times U_+$. In the finite-dimensional situation this is achieved in [Kos73].

4C Reflections, transvections and reflection-homogeneity

Since θ fixes both $C_K(G)$ and K , it induces an involutive automorphism of G_{eff} and an involutive permutation $\theta : G/K \rightarrow G/K$ via $\theta(gK) := \theta(g)K$. Defining the basepoint of G/K as $o := eK$, one in fact has $s_o(gK) = \tau(e)\theta(g)K = \theta(gK)$, i.e., θ coincides with the point reflection s_o of the symmetric space G/K at o . In particular, one obtains a subgroup

$$G_{\text{eff}} \rtimes \langle \theta \rangle < \text{Aut}(G/K, \mu).$$

Similarly, by Proposition 3.16 the Cartan–Chevalley involution θ induces an involution $\overline{\theta} : \overline{G} \rightarrow \overline{G}$ which in turn yields an involutive automorphism $\overline{\theta} : \overline{G}/\overline{K} \rightarrow \overline{G}/\overline{K}$ and a subgroup

$$\text{Ad}(G) \rtimes \langle \overline{\theta} \rangle < \text{Aut}(\overline{G}/\overline{K}, \overline{\mu}),$$

where $\overline{\theta}$ corresponds to the point reflection at $\overline{o} := e\overline{K}$.

Proposition 4.8. *The set of point reflections of G/K (respectively, $\overline{G}/\overline{K}$) equals the conjugacy class of s_o (respectively, $s_{\overline{o}}$) in $G_{\text{eff}} \rtimes \langle \theta \rangle$ (respectively, $\text{Ad}(G) \rtimes \langle \overline{\theta} \rangle$). Moreover, the set of transvections of G/K (respectively, $\overline{G}/\overline{K}$) is given by $\tau(G)^2 C_K(G)$ (respectively, $\overline{\tau}(\overline{G})^2 Z(\overline{G})$). Furthermore, the respective transvection groups of G/K and $\overline{G}/\overline{K}$ are*

$$\text{Trans}(G/K, \mu) = G_{\text{eff}} \quad \text{and} \quad \text{Trans}(\overline{G}/\overline{K}, \overline{\mu}) = \text{Ad}(G).$$

The main groups of G/K , respectively $\overline{G}/\overline{K}$ are given by

$$G(G/K, \mu) = G_{\text{eff}} \rtimes \langle \theta \rangle \quad \text{and} \quad G(\overline{G}/\overline{K}, \overline{\mu}) = \text{Ad}(G) \rtimes \langle \overline{\theta} \rangle.$$

Finally, G/K and $\overline{G}/\overline{K}$ are reflection-homogeneous.

Proof. For $g, h \in G$ one has

$$\begin{aligned} s_{gK}(hK) &= \mu(gK, hK) = \tau(g)\theta(h)K = g\theta(g)^{-1}\theta(h)K \\ &= g\theta(g^{-1}h)K = (gC_K(G) \circ s_o \circ g^{-1}C_K(G))(hK) \\ &= (gC_K(G) \circ \theta \circ g^{-1}C_K(G))(hK), \end{aligned} \tag{4.3}$$

i.e., s_{gK} is conjugate to s_o via $gC_K(G) \in G_{\text{eff}}$. Furthermore, observe that for $g \in G$ one has

$$s_{gK} \circ s_o = gC_K(G) \circ s_o \circ g^{-1}C_K(G) \circ s_o = g\theta(g)^{-1}C_K(G) = \tau(g)C_K(G), \tag{4.4}$$

Given $g, h \in G$ therefore

$$s_{gK}s_{hK} = (s_{gK}s_o)(s_{hK}s_o)^{-1} = \tau(g)\tau(h)^{-1}C_K(G) = \tau(g)\tau(\theta(h))C_K(G),$$

whence the transvections are exactly the elements of $\tau(G)^2 C_K(G) \supset \tau(G)C_K(G)$. The other claims concerning G now follow readily, using Proposition 3.30 and Lemma 2.7. The claims concerning \overline{G} are shown analogously. \square

4D Models for Kac-Moody symmetric spaces

Recall from Section 2A that every reflection-homogeneous symmetric space can be realized as a subset of its main group (the “involution model” from Lemma 2.7) and as a subset of its transvection group (the “quadratic representation” from Remark 2.8) with suitably defined multiplications.

In view of Example 2.6 and Proposition 4.8 the *involution model* of the reflection-homogeneous symmetric space $(G/K, \mu)$ is given by the pair $(\mathcal{X}, \hat{\mu})$ where

$$\mathcal{X} := \{ {}^g\theta \in G_{\text{eff}} \rtimes \langle \theta \rangle \mid g \in G_{\text{eff}} \rtimes \langle \theta \rangle \} \quad \text{and} \quad \hat{\mu} : \mathcal{X} \times \mathcal{X} \rightarrow \mathcal{X}, \quad (\alpha, \beta) \mapsto \alpha\beta\alpha.$$

The map $\pi : G \rightarrow \mathcal{X}, \quad g \mapsto {}^g\theta = gC_K(G) \circ \theta \circ g^{-1}C_K(G)$ by (4.3) factors through $\hat{\pi} : G/K \rightarrow \mathcal{X}$, which is an isomorphism of reflection spaces.

The *quadratic representation* of G/K depends on the choice of a basepoint $o \in G/K$. For $o = eK$ by Proposition 4.8 the quadratic representation is given by the map

$$t : G/K \rightarrow G_{\text{eff}}, \quad gK \mapsto s_{gK} \circ s_o.$$

By (4.4) one has $s_{gK} \circ s_o = \tau(g)C_K(G)$. Thus the image $\mathcal{T} = T(G/K, \mu, o) \subset \text{Trans}(G/K, \mu)$ of the quadratic representation of G/K is given by the image of $\tau(G)$ in G_{eff} , and the product on \mathcal{T} is given by $\tilde{m}(s, t) := st^{-1}s$ by Remark 2.8. Note that τ induces an isomorphism of reflection spaces $(G/K, \mu) \rightarrow (\mathcal{T}, \tilde{m})$.

By definition, the canonical projection $G \rightarrow G_{\text{eff}}$ restricts to a surjection $\tau(G) \rightarrow \mathcal{T}$. Since the kernel of the projection $G \rightarrow G_{\text{eff}}$ is contained in K , it intersects $\tau(G)$ trivially by Proposition 3.29. It follows that the projection $\tau(G) \rightarrow \mathcal{T}$ is actually bijective and so by transport of structure the multiplication

$$\tilde{\mu} : \tau(G) \times \tau(G) \rightarrow \tau(G), \quad \tilde{\mu}(x, y) = xy^{-1}x$$

provides a symmetric space such that

$$(\tau(G), \tilde{\mu}) \cong (\mathcal{T}, \tilde{m}) \cong (G/K, \mu).$$

This symmetric space $(\tau(G), \tilde{\mu})$ is called the *group model* of G/K .

The left-multiplication action of G on G/K translates into G -actions on \mathcal{T} and $\tau(G)$ by automorphisms. Since $t(ghK) = \tau(gh)C_K(G) = g * \tau(h)C_K(G)$, the induced G -action on $\tau(G)$ is given by twisted conjugation. It follows that the isomorphism $G/K \rightarrow \tau(G)$ is explicitly given by

$$\hat{\tau} : G/K \rightarrow \tau(G) : gK = geK \mapsto g * \tau(e) = \tau(g).$$

Combining the isomorphisms $\hat{\tau} : G/K \rightarrow \tau(G)$ and $\hat{\pi} : G/K \rightarrow \mathcal{X}$ one also obtain an isomorphism $\rho : \tau(G) \rightarrow \mathcal{X}$ making the diagram in Figure 2 commute. Denoting by $[h]$ the image of $h \in G$ under the projection $G \rightarrow G_{\text{eff}}$ this isomorphism is explicitly given as follows.

Lemma 4.9. *Let $\rho : \tau(G) \rightarrow \mathcal{X} : h \mapsto [h]\theta$. Then ρ makes the diagram in Figure 2 commute. In particular, it is an isomorphism of reflection spaces.*

Proof. It suffices to check that $\rho \circ \tau = \pi$. For this one computes

$$\rho \circ \tau(g) = \rho(g\theta(g)^{-1}) = [g\theta(g)^{-1}]\theta = [g]\theta[g]^{-1}. \quad \square$$

Remark 4.10. For each of the three models of the unreduced symmetric space there is a corresponding model of the reduced symmetric space. The *coset model* $\overline{G}/\overline{K}$ was already discussed above. The *involution model* of $\overline{G}/\overline{K}$ is given by the conjugacy class $\overline{\mathcal{X}}$ of $s_{\overline{\sigma}} = \overline{\theta}$ in $\text{Ad}(G) \rtimes \langle \overline{\theta} \rangle$. Since the latter group can be embedded as a subgroup into the automorphism groups $\text{Aut}(G) \subset$

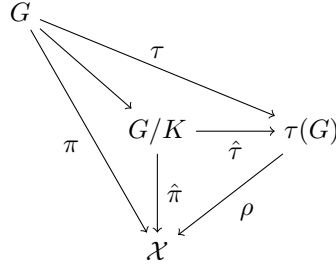


Figure 2: Isomorphisms between the different models.

$\text{Aut}(\Delta)$ of the group G and⁴ its twin building Δ , one can consider $\overline{\mathcal{X}}$ both as a set of involutions of the group G and of the twin building Δ . In either of these pictures, the multiplication is given by

$$\widehat{\mu}(\alpha, \beta) = \alpha \circ \beta^{-1} \circ \alpha.$$

The *group model* of $\overline{G}/\overline{K}$ is given by $(\overline{\tau}(\overline{G}), \widehat{\mu})$ with multiplication given by

$$\widehat{\mu}(x, y) = xy^{-1}x.$$

As in the unreduced model one has isomorphism between these models as depicted in Figure 3. Here the isomorphism $\overline{\rho} : \overline{\tau}(\overline{G}) \rightarrow \overline{\mathcal{X}} \subset \text{Aut}(G)$ is given by

$$\overline{\rho}(g) = c_g \circ \theta, \quad (4.5)$$

where c_g denotes the inner automorphism defined by g .

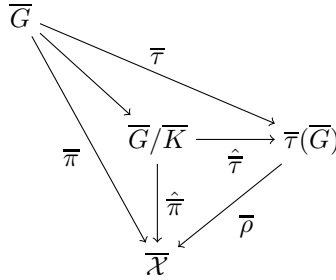


Figure 3: Isomorphisms between the reduced models.

4E Comparison of topologies

Sections 4C and 4D provided three mutually isomorphic models of the reduced Kac–Moody symmetric space — the coset model $(\overline{G}/\overline{K}, \overline{\mu})$, the involution model $(\overline{\mathcal{X}}, \widehat{\mu})$, and the group model $(\overline{\tau}(\overline{G}), \widehat{\mu})$. Each of these models comes equipped with a canonical topology.

In this section we prove Proposition 1.5.

Definition 4.11. Topologize the sets $\overline{G}/\overline{K}$, $\overline{\tau}(\overline{G})$, $\overline{\mathcal{X}}$ as follows:

- (i) $\overline{G}/\overline{K}$ is endowed with the quotient topology from the projection map $\overline{G} \rightarrow \overline{G}/\overline{K}$.

⁴See Corollary 6.3 below for the fact that $\text{Aut}(G)$ embeds into $\text{Aut}(\Delta)$.

- (ii) $\tau(\overline{G}) \subset \overline{G}$ inherits the subspace topology from \overline{G} .
- (iii) $\text{Ad}(G)$ carries the quotient topology as a quotient of G or, equivalently, \overline{G} , the semi-direct product $\text{Ad}(G) \rtimes \langle \theta \rangle$ is endowed with the unique group topology in which the finite index subgroup $\text{Ad}(G)$ is open and carries that quotient topology, and finally $\overline{\mathcal{X}} \subset \text{Ad}(G) \rtimes \langle \theta \rangle$ inherits the subspace topology.

Theorem 4.12. *The maps $\hat{\pi}$ and $\hat{\tau}$ in Figure 3 are continuous and the map $\bar{\rho}$ in Figure 3 is a homeomorphism. If the generalized Cartan matrix \mathbf{A} is two-spherical, then each of the maps $\hat{\pi}$, $\hat{\tau}$ and $\bar{\rho}$ in Figure 3 is a homeomorphism; in particular, the topological symmetric spaces $(\overline{G}/\overline{K}, \bar{\mu})$, $(\overline{\mathcal{X}}, \bar{\mu})$ and $(\tau(\overline{G}), \bar{\mu})$ are mutually isomorphic.*

Proof. By the commuting diagram in Figure 3 it suffices to investigate the maps $\hat{\tau}$ and $\bar{\rho}$. Since \widehat{G} is a topological group, Lemma 4.9 implies that $\bar{\rho}$ is continuous, so that it remains to show that $\bar{\rho}$ is open. Proposition 3.29 and Theorem 3.23(i) imply $\tau(\overline{G}) \cap Z(\overline{G}) \leq \tau(\overline{G}) \cap \overline{K} = \{e\}$. One concludes that $\tau(\overline{G})$ embeds into $\text{Ad}(G)$. After identifying $\tau(\overline{G})$ with its image in $\text{Ad}(G)$ according to (4.5) the map $\bar{\rho}^{-1} : \mathcal{X} \rightarrow \tau(G)$ is given by $\psi \mapsto \psi \circ \theta^{-1}$. Since $\text{Ad}(G) \rtimes \langle \theta \rangle$ is a topological group, $\bar{\rho}^{-1}$ is continuous, and hence $\bar{\rho}$ is open, i.e., a homeomorphism.

The map $\hat{\tau}$ is continuous, since the twist map is continuous. For the openness of $\hat{\tau}$ note that by the topological Iwasawa decomposition Theorem 3.23 and, if \mathbf{A} is two-spherical, by Corollary 3.32 there are homeomorphisms $h_1 : U_+ \times \overline{A} \rightarrow \overline{G}/\overline{K} : (u_+, a) \mapsto u_+ a \overline{K}$ and $h : \tau(\overline{G}) \rightarrow U_+ \times \overline{A}$ with $h^{-1}(u_+, a) = u_+ a \theta(u_+)^{-1}$. It thus suffices to show that the composition

$$h_2 : U_+ \times \overline{A} \xrightarrow{h_1} \overline{G}/\overline{K} \xrightarrow{\hat{\tau}} \tau(G) \xrightarrow{h} U_+ \times \overline{A}$$

is open. Now $\hat{\tau} \circ h_1(u_+, a) = u_+ a^2 \theta(u_+)^{-1}$ and, hence, $h_2(u_+, a) = (u_+, a^2)$. Thus the theorem follows from the fact that the map $\overline{A} \rightarrow \overline{A} : a \mapsto a^2$ is open. \square

Remark 4.13. Once the issue discussed in Remark 3.13 has been resolved, the preceding theorem will automatically hold without the requirement that \mathbf{A} be two-spherical.

5 Flats and geodesics in Kac–Moody symmetric spaces

Throughout this section G denotes a simply connected centered split real Kac–Moody group satisfying the assumptions of Convention 4.2 on page 33, the group \overline{G} denotes its semisimple adjoint quotient, and $\text{Ad}(G)$ its adjoint quotient. Moreover, $\Delta = \Delta^- \sqcup \Delta^+$ denotes the common twin building associated to the canonical BN pairs of these groups.

The purpose of this section is to investigate the flats of the Kac–Moody symmetric spaces G/K and $\overline{G}/\overline{K}$.

5A Standard flats

We start by constructing explicit examples of Euclidean flats in Kac–Moody symmetric spaces. We will see in Theorem 5.15 below that these are exactly the maximal flats. Recall from Proposition 3.8(i) that we have homeomorphisms $\exp : \mathfrak{a} \rightarrow A$ and $\exp : \overline{\mathfrak{a}} \rightarrow \overline{A}$.

Proposition 5.1. *Equip \mathfrak{a} (respectively $\overline{\mathfrak{a}}$) with its Euclidean reflection space structure. Then for every $g \in G$ (respectively, $\overline{g} \in \overline{G}$) the map*

$$\varphi_g : \mathfrak{a} \rightarrow gAK, \quad X \mapsto g \exp(X)K \quad (\text{respectively, } \varphi_{\overline{g}} : \overline{\mathfrak{a}} \rightarrow \overline{g}\overline{A}\overline{K}, \quad X \mapsto \overline{g} \exp(X)\overline{K})$$

is an isomorphism of topological reflection spaces. Moreover, the subset $gAK \subset G/K$ (respectively, $\overline{g}\overline{A}\overline{K} \subset \overline{G}/\overline{K}$) is closed, hence a Euclidean flat of dimension $\dim \mathfrak{a} = n$ (respectively, $\dim \overline{\mathfrak{a}} = \text{rk}(\mathbf{A})$).

Proof. First observe that the subsets $gAK \subset G/K$ (respectively, $\bar{g}\bar{A}\bar{K} \subset \bar{G}/\bar{K}$) are closed. By Theorem 3.23, multiplication $\bar{U}_\pm \times \bar{A} \times \bar{K} \rightarrow G$ induces a homeomorphism. Therefore, $\bar{A}\bar{K}$ and any of its translates $\bar{g}\bar{A}\bar{K}$ are closed in \bar{G}/\bar{K} , and so are the preimages gAK in G/K . It remains to show that the maps φ_g are isomorphisms of reflection spaces. Since both G and \bar{G} act by automorphisms, one may assume that $g = e$, respectively $\bar{g} = e$. Thus let $X, Y \in \mathfrak{a}$. Using that $\theta(t) = t^{-1}$ for all $t \in A = \tau(A)$ (see Lemmas 3.18 and 3.20) and that \exp is a group homomorphism one computes

$$\begin{aligned} \mu(\varphi_e(X), \varphi_e(Y)) &= \tau(\exp(X))\theta(\exp(Y))K \\ &= \exp(X)\theta(\exp(X))^{-1}\theta(\exp(Y))K \\ &= \exp(X)\exp(X)\exp(-Y)K \\ &= \exp(2X - Y)K \\ &= \varphi_e(X \cdot Y), \end{aligned}$$

and the computation for the reduced case is identical. \square

Definition 5.2. For every $g \in G$ (respectively, $\bar{g} \in \bar{G}$) the flat $gAK \subset G/K$ (respectively, $\bar{g}\bar{A}\bar{K} \subset \bar{G}/\bar{K}$) is called a *standard flat*.

The following proposition describes images of standard flats under the various isomorphisms of models. By abuse of language we will also refer to these images as standard flats in the respective models.

Proposition 5.3. (i) The image of the standard flat gAK under the isomorphism $\hat{\pi} : G/K \rightarrow \mathcal{X}$ is given by

$$\mathcal{X}_{gT} := \{\alpha \in \mathcal{X} \mid \alpha(^gT) \subseteq ^gT\} = \{\alpha \in \mathcal{X} \mid \alpha(^gT) = ^gT\}.$$

(ii) The image of the standard flat gAK under the isomorphism $\hat{\tau} : G/K \rightarrow \tau(G)$ is given by

$$F[g] := g * A = g * \tau(A) \subset \tau(G).$$

The analogous statements hold for G replaced by \bar{G} .

Proof. Observe that the two descriptions of \mathcal{X}_{gT} indeed coincide because \mathcal{X} consists of involutions. Moreover, the maps $g \mapsto gAK$ and $g \mapsto \mathcal{X}_{gT}$ and $g \mapsto F[g]$ are all equivariant under the respective G -actions. It therefore suffices to show that

$$\hat{\pi}(AK) = \mathcal{X}_T \quad \text{and} \quad \hat{\tau}(TK) = A. \quad (5.1)$$

Certainly, $\hat{\pi}(AK) \subseteq \pi(T) \subseteq \mathcal{X}_T$. Conversely, let ${}^h\theta \in \mathcal{X}_T$. This means that

$$T = {}^h\theta(T) = (h \circ \theta \circ h^{-1})(T) = h\theta(h^{-1}Th)h^{-1} = \tau(h)\theta(T)\tau(h)^{-1} = \tau(h)T\tau(h)^{-1}.$$

Hence $\tau(h) \in N_G(T)$. By Corollary 3.24 and Lemma 3.20 one has $N_G(T) \cap \tau(G) = A = \tau(A)$, so there is $t \in A$ such that $\tau(h) = \tau(t)$ and, therefore, $tK = hK$ by Lemma 3.18. Thus $hK = tK \in AK$, showing that ${}^h\theta = {}^t\theta = \hat{\pi}(tK) \in \hat{\pi}(AK)$ and hence

$$\hat{\pi}(AK) = \pi(T) = \mathcal{X}_T.$$

Finally, $\hat{\tau}(TK) = \tau(T) = A$. This establishes (5.1) and finishes the proof. \square

Remark 5.4. Denote by $\mathcal{F}_{\text{std}}(G/K)$ the set of standard flats in G/K . By definition, G acts transitively on $\mathcal{F}_{\text{std}}(G/K)$ via left-multiplication. Recall from Lemma 3.21 that $N_G(T) = A \rtimes N_K(T)$. Since A is the identity component of T (see Definition 3.5), one has $N_K(T) \leq N_K(A)$, since conjugation in G is continuous. Conversely, by [Cap09, Lemma 4.9] the torus T is the unique torus of

G containing A , so any element normalizing A necessarily has to normalize T , and one deduces that

$$N_K(A) = N_K(T). \quad (5.2)$$

Thus every $g = N_G(T)$ can be written as $g = ak$ with $a \in A$ and $k \in N_K(A)$, and thus $gAK = akAK = a(kAk^{-1})kK = aAK = AK$. In other words, $N_G(T)$ stabilizes AK .

The coset space $G/N_G(T)$ can be identified with the set $\mathcal{T}(G)$ of maximal tori of G via the map $gN_G(T) \mapsto {}^gT$. One thus obtains a G -equivariant surjection

$$\mathcal{T}(G) \rightarrow \mathcal{F}_{\text{std}}(G/K), \quad {}^gT \mapsto gAK. \quad (5.3)$$

In other words, the maximal standard flats are parametrized by the maximal tori. The same argument applies to \overline{G} instead of G .

Assertion (ii) of Proposition 5.3 implies that the parametrization map in (5.3) is actually a bijection: Indeed, the standard flat associated with gT in the group model is given by $F[g] = g * A = gA\theta(g)^{-1}$, and one has

$$F[g]\theta(F[g]) = gA\theta(A)g^{-1} = gAg^{-1}.$$

One can therefore recover gAg^{-1} from the associated flat. Now by [Cap09, Lemma 4.9] the group gAg^{-1} is contained in a unique maximal torus of G , and this maximal torus is exactly gT . Thus $F[g]$ determines gT , and the map (5.3) is thus bijective.

The same argument applies to maximal tori in \overline{G} , as $C < T$ (cf. Definition 3.5) is central in G , whence contained in any G -conjugate of T and, moreover, stabilized by any conjugate of θ .

Note that maximal tori in G are precisely the chamberwise stabilizers of the twin apartments of the twin building Δ , as are the maximal tori in \overline{G} . Altogether one observes the following:

Corollary 5.5. *The following objects are in G -equivariant bijection with the elements of $G/N_G(T) = \overline{G}/N_{\overline{G}}(\overline{T})$:*

- (i) *twin apartments of Δ ,*
- (ii) *maximal tori of \overline{G} ,*
- (iii) *maximal tori of G ,*
- (iv) *standard flats in G/K ,*
- (v) *standard flats in $\overline{G}/\overline{K}$.*

In particular, G acts transitively on these objects, every standard flat in G/K projects to a standard flat in $\overline{G}/\overline{K}$, and every standard flat in $\overline{G}/\overline{K}$ lifts uniquely to a standard flat in G/K . \square

By Theorem 5.15 below the standard flats in either of the two Kac–Moody symmetric spaces are exactly the maximal flats. This in turn implies that the maximal flats in $\overline{G}/\overline{K}$ are in one-to-one correspondence to the maximal flats in G/K .

Remark 5.6. By definition, if F is a standard flat in G/K , then there exists an automorphism α of $\overline{\mathcal{X}}$ (in fact, even an element $g \in G$) mapping the flat AK to F . Since A acts transitively on AK , even for every pair (p, F) consisting of a standard flat F and a point $p \in F$ there exists an automorphism α of $\overline{\mathcal{X}}$ which maps AK to F and 0 to p . If α is any such automorphism, then by Proposition 5.1 there exists an isomorphism of topological reflection spaces given by

$$\varphi : \mathfrak{a} \rightarrow F, \quad X \mapsto \alpha(\exp(X)K)$$

which maps 0 to p . In the sequel we refer to any isomorphism arising in this way from an automorphism of $\overline{\mathcal{X}}$ as a *chart* of F centered at p .

Similarly, one defines charts of pointed standard flats in $\overline{G}/\overline{K}$. Also define a chart of a pointed standard flat in one of the other models as a composition of a charts in the coset model with the respective isomorphism of models.

5B Midpoint convex subsets and geodesic connectedness

Our next goal is to characterize midpoint convex subsets of Kac–Moody symmetric spaces. The following definition borrowed from [Cap09, Section 4.2.2] is key to this characterization.

Definition 5.7. An element $g \in G$ (or $\bar{g} \in \bar{G}$) is called *diagonalizable* if it stabilizes a pair of opposite chambers in Δ and, hence, stabilizes a twin apartment chamberwise.

The following example shows that, in the non-spherical case, elements of $\bar{\tau}(\bar{G})$ need not be diagonalizable. The reader is referred to [Hor17] for a more detailed discussion of this theme.

Example 5.8. Let $n \geq 1$ and consider the affine example $\bar{G} := \mathrm{SL}_{n+1}(\mathbb{R}[t, t^{-1}])$ of type \tilde{A}_n with the Cartan–Chevalley involution $\theta(x) := ((x^{-1})^T)^\sigma$, where σ is the ring automorphism of $\mathbb{R}[t, t^{-1}]$ which fixes \mathbb{R} and interchanges t and t^{-1} . Then let

$$\begin{aligned} u &:= \begin{pmatrix} 1 & 1+t & & \\ 0 & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \in B_+, & v &:= \tau(u) = u\theta(u)^{-1} = \begin{pmatrix} 1 & 1+t & & \\ 0 & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & & \\ 1+t^{-1} & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \\ & & & = \begin{pmatrix} 1+(1+t)(1+t^{-1}) & 1+t & & \\ 1+t^{-1} & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \end{aligned}$$

and the characteristic polynomial of v is

$$\begin{aligned} c_\lambda(v) &= ((\lambda - (1 + (1+t)(1+t^{-1}))) (\lambda - 1) - (1+t)(1+t^{-1})) \cdot (\lambda - 1)^{n-1} \\ &= (\lambda^2 - (t + 4 + t^{-1})\lambda + 1) \cdot (\lambda - 1)^{n-1}. \end{aligned}$$

However, the polynomial $c_\lambda(v)$ does not split into linear factors over $\mathbb{R}[t, t^{-1}]$, whence v is not conjugate within \bar{G} to an element of the torus \bar{T} , which consists of diagonal matrices with entries from \mathbb{R} .

The following result demonstrates that the behaviour described in the preceding example is not merely an affine but instead a general non-spherical phenomenon:

Theorem 5.9 ([Hor17, Theorem 5.6]). *The set $Q := \bigcap_{i=1}^\infty \tau^i(G)$ (respectively, $\bar{Q} := \bigcap_{i=1}^\infty \bar{\tau}^i(\bar{G})$) equals the set of diagonalizable elements in $\tau(G)$ (respectively, $\bar{\tau}(\bar{G})$). Moreover, if G is of non-spherical type, then $Q \neq \tau(G)$ and $\bar{Q} \neq \bar{\tau}(\bar{G})$, i.e. both $\tau(G)$ and $\bar{\tau}(\bar{G})$ contain elements which are not diagonalizable.*

The description of the set of diagonalizable elements in $\tau(G)$, respectively $\bar{\tau}(\bar{G})$ has the following implication:

Corollary 5.10. *If $F \subseteq \tau(G)$ (or $F \subseteq \bar{\tau}(\bar{G})$) is midpoint convex and $e \in F$, then any $x \in F$ is diagonalizable.*

Proof. Let $x \in F$. Then, by midpoint convexity, there is $x' \in F$ such that $x = s_{x'}(e) = \tilde{\mu}(x', e) = x'^2 = \tau(x')$, where $\tilde{\mu}$ is the multiplication map of the group model from Section 4D and the last equality holds by Lemma 3.18(i). Iteration of this argument implies that for every $n \in \mathbb{N}$ there is $z_n \in F$ such that $z_n^{2^n} = \tau^n(z_n) = x$. Hence $x \in \bigcap_{i=1}^\infty \tau^i(G)$ (respectively, $x \in \bigcap_{i=1}^\infty \bar{\tau}^i(\bar{G})$) and, thus, x is diagonalizable by the preceding theorem. \square

Corollary 5.11. *In every non-spherical Kac–Moody symmetric space (reduced or unreduced) there exists a pair of points that do not admit a midpoint and therefore do not lie on a common geodesic.* \square

For instance, the elements $\mathrm{id} = \tau(\mathrm{id})$ and $\tau(u)$ from Example 5.8 do not admit a midpoint.

Remark 5.12. The preceding corollary illustrates that Kac–Moody symmetric spaces suffer from exactly the same deficits as the *masures* introduced in [GR08, Section 3] and discussed in detail in [Rou11].

Despite the lack of geodesics expressed by Corollary 5.11 one nevertheless has the following:

Proposition 5.13 (cf. [Hor17, Proposition 6.5]). *Kac–Moody symmetric spaces are geodesically connected. In particular,*

$$G = \bigcup_{n \in \mathbb{N}} (KAK)^n.$$

Proof. One needs to show that any pair $x, y \in \tau(G)$ can be connected by a piecewise geodesic curve. The resulting geodesic connectedness of G/K then implies that of $\overline{G}/\overline{K}$.

By transitivity of the action of the group G on the symmetric space $\tau(G)$ one may assume without loss of generality that $x = e$. By Proposition 3.31(iii) one can write $y = \tau(u_1 \cdots u_k t)$ with $t \in A$ and $u_i \in U_{\beta_i}$ for some $\beta_i \in \Phi^+$.

For $\alpha \in \Phi^+$ and $u \in U_\alpha$ and $t \in A$, the element

$$\tau(ut) = ut\theta(t)^{-1}\theta(u)^{-1} = ut^2\theta(u)^{-1} \in A\langle U_\alpha, U_{-\alpha} \rangle$$

(cf. Lemma 3.20) stabilizes two opposite spherical residues (in fact, two opposite panels), whence is diagonalizable by Lemma 3.22. Applying this to $u_i t_i$ with $t_i = 1$ for $1 \leq i < k$ and $t_k = t$, one obtains standard flats F'_i containing e and $\tau(u_i t_i)$ and, thus, geodesic segments joining e and $\tau(u_i t_i)$. Then $F_i := u_1 t_1 \cdots u_{i-1} t_{i-1} * F'_i$ is a standard flat containing $\tau(u_1 t_1 \cdots u_{i-1} t_{i-1})$ and $\tau(u_1 t_1 \cdots u_i t_i)$. Setting $x_0 = e$ and $x_i := \tau(u_1 t_1 \cdots u_i t_i)$ for $0 \leq i \leq k$, one has $x_i \in F_i \cap F_{i+1}$ and, moreover, $x_k = y$. The claim follows. \square

We have proved Theorem 1.8.

Remark 5.14. Note that in the proof of the preceding proposition one actually has quite some freedom in choosing the individual geodesic segments. For instance, by (RGD6) for any factorization $t = t_1 \cdots t_k$ within A there exist $u'_i \in U_{\beta_i}$ such that

$$u_1 \cdots u_k t = u'_1 t_1 \cdots u'_k t_k.$$

Of course, the argument in the proof applies to any such factorization. This observation plays a crucial role in Section 8 below.

5C The classification of maximal flats

The methods for analyzing flats developed so far allow one to characterize the maximal (weak) flats in Kac–Moody symmetric spaces as follows. The proof of the following theorem makes use of the various different models of the Kac–Moody symmetric space, in particular the group model. Here, the topology on the group model is the one transported from the coset model G/K via the bijection $\hat{\tau}$; by Theorem 4.12 this topology coincides with the natural topology of the group model in the two-spherical case.

Theorem 5.15. *Every weak flat in a Kac–Moody symmetric space (reduced or unreduced) is contained in a standard flat. In particular,*

- (i) *standard flats are exactly the maximal weak flats;*
- (ii) *all weak flats are Euclidean;*
- (iii) *G , respectively \overline{G} acts transitively on maximal flats.*

Proof. Let $F \subset \tau(G)$ be a weak flat. It suffices to show that F is contained in a standard flat corresponding to some maximal torus of G . Since G acts transitively on $\tau(G)$, one may additionally assume without loss of generality that $e \in F$. Note that this assumption will in fact enable us to prove that the flat F is contained in a standard flat of a θ -split maximal torus, i.e., that θ acts by inversion on that maximal torus.

From now on assume $e \in F$, let $x, y \in F$, and use the notation for the group model from Section 4D.

Claim 1. $[xy, yx] = e$, or equivalently, $xy^2x = yx^2y$.

One computes

$$\tilde{\mu}(x, \tilde{\mu}(e, \tilde{\mu}(y, e))) = \tilde{\mu}(x, \tilde{\mu}(e, y^2)) = \tilde{\mu}(x, y^{-2}) = xy^2x$$

and, similarly,

$$\tilde{\mu}(y, \tilde{\mu}(e, \tilde{\mu}(x, e))) = yx^2y.$$

Hence, as F is weakly abelian,

$$\begin{aligned} xy^2x &= \tilde{\mu}(x, \tilde{\mu}(e, \tilde{\mu}(y, e))) = \tilde{\mu}(y, \tilde{\mu}(e, \tilde{\mu}(x, e))) = yx^2y \\ \Leftrightarrow (xy)(yx) &= (yx)(xy) \\ \Leftrightarrow [xy, yx] &= e. \end{aligned}$$

Claim 2. xy is diagonalizable.

By midpoint convexity of F , there is a midpoint $x' \in F$ between e and x , whence $x'^2 = s_{x'}(e) = x$. Moreover, $s_e(y) = y^{-1} \in F$, and so $s_{x'}(y^{-1}) = x'yx' \in F$. By Corollary 5.10 the element $x'yx'$ is diagonalizable. Hence, by definition, there exists a twin apartment Σ of the twin building of G which is fixed chamberwise by $x'yx'$. Let c be a chamber of Σ and set $(\Sigma', c') := x'.(\Sigma, c)$. Then

$$y.(\Sigma', c') = yx'.(\Sigma, c) = x'^{-1}(x'yx').(\Sigma, c) = x'^{-1}(\Sigma, c) = x'^{-2}x'(\Sigma, c) = x^{-1}.(\Sigma', c'),$$

and so xy stabilizes Σ' and fixes c' . That is, xy fixes Σ' pointwise and, by definition, is diagonalizable.

Claim 3. In each half Δ_{\pm} of the twin building there exist opposite spherical residues $R_+ \subset \Delta_+$ and $R_- = \theta(R_+) \in \Delta_-$ stabilized by both xy and yx .

Both xy and $yx = \theta(y)^{-1}\theta(x)^{-1} = \theta(xy)^{-1}$ (by Lemma 3.18(i) plus $x, y \in F \subset \tau(G)$) are diagonalizable and, thus, both fix some twin apartment chamberwise. In particular, both admit fixed points in the CAT(0) realizations X_{\pm} of either half Δ_{\pm} of the twin building. (See [Dav98], also [Cap09, Section 2.1].)

One can now find a common fixed point of xy and yx in X_+ by a standard commutation argument as follows: For $p \in \text{Fix}(xy)$, one has

$$yx.p = yx.(xy.p) \stackrel{\text{Claim 1}}{=} xy.(yx.p),$$

whence $p.yx \in \text{Fix}(xy)$. Thus the convex set $\text{Fix}(xy)$ is preserved by the isometry yx . Let q be a point fixed by yx and r_+ its (unique) projection to $\text{Fix}(xy)$ in the CAT(0) space X_+ . Since $\text{Fix}(xy)$ is preserved by yx , it follows that r_+ is also fixed by yx . The point $r_+ \in X_+$ corresponds to a spherical residue R_+ of Δ_+ stabilized by both yx and xy .

Consequently, the residue $R_- := \theta(R_+)$ opposite R_+ is stabilized by both $\theta(xy) = (yx)^{-1}$ and $\theta(yx) = (xy)^{-1}$ and, hence, also by $yx \in \langle (yx)^{-1} \rangle < G$ and $xy \in \langle (xy)^{-1} \rangle < G$.

Claim 4. xy fixes a chamber $d \in R_+$ and yx fixes $\hat{d} := \text{proj}_{R_+} \theta(d)$ opposite d in R_+ .

Since xy is diagonalizable, it fixes a twin apartment chamberwise and so there is a chamber $c \in \Delta_+$ fixed by xy . Thus the chamber $d := \text{proj}_{R_+}(c)$ is also fixed by xy . The involution θ induces an involution $\theta_{R_+}(c) := \text{proj}_{R_+}(\theta(c))$ on R_+ , which maps every chamber in R_+ to a chamber opposite in R_+ . The chamber $\hat{d} := \theta(d)$ is fixed by $yx = \theta(xy)^{-1}$:

$$yx.\hat{d} = \text{proj}_{yx.R_+}(yx.\theta(d)) \stackrel{\text{Claim 3}}{=} \text{proj}_{R_+}(yx.\theta(d)) = \text{proj}_{R_+}(\theta((xy)^{-1}.d)) = \text{proj}_{R_+}(\theta(d)) = \hat{d}.$$

Claim 5. *There exists a chamber $d' \in R_+$ fixed by both xy and yx .*

By Claims 3 and 4 the elements xy and yx are contained in opposite Borel subgroups of the reductive split real Lie group stabilizing the opposite spherical residues R_+ and $\theta(R_+)$ (cf. [HKM13, Corollary 7.16]).

This reductive Lie group is a subgroup of $\mathrm{GL}_{n+1}(\mathbb{R})$. By [HN12, Proposition 16.1.5] one can model the stabilizer of d as lower triangular matrices, the stabilizer of \hat{d} as upper triangular matrices, and θ as transpose-inverse. Thus $yx = \theta(xy)^{-1} = (xy)^T$. One concludes that both xy and yx are diagonal in this coordinatization: Suppose

$$xy = \begin{pmatrix} v_1 & v_2 & \dots & v_n \\ 0 & * & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & * \end{pmatrix} \quad \text{and thus} \quad yx = (xy)^T = \begin{pmatrix} v_1 & 0 & \dots & 0 \\ v_2 & * & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ v_n & * & \dots & * \end{pmatrix}.$$

Computing the product $xy \cdot yx$ yields the top left entry $v_1^2 + \dots + v_n^2$. On the other hand, the top left entry of the $yx \cdot xy$ is v_1^2 . By Claim 1 one has $[xy, yx] = e$ and hence $v_2^2 + \dots + v_n^2 = 0$ and so $v_2 = \dots = v_n = 0$. Inductively one obtains that xy and yx act by the same diagonal matrix on R_+ . Therefore there is a chamber d' stabilized by both xy and yx .

Claim 6. $xy = yx$.

Since xy and yx stabilize a chamber d' , one has $\theta(d') = \theta(xy.d') = x^{-1}y^{-1}.\theta(d')$, thus $\theta(d') = yx.\theta(d')$. It follows that xy and yx stabilize both d' and $\theta(d')$ and, hence, fix a θ -stable twin apartment. Thus they are contained in a common θ -split torus. As an immediate consequence, $(xy)^{-1} = \theta(xy) = \theta(x)\theta(y) = x^{-1}y^{-1}$. Hence $xy = yx$.

Claim 7. *For each $x, y \in F$ one has $y^{-1}, xy \in F$. That is, F is a commutative subgroup of G .*

Recall that $e \in F$ by assumption. Let $x' \in F$ be a midpoint of x and y . Then $s_e(y) = y^{-1} \in F$ and $s_{x'}(y^{-1}) = x'yx' \stackrel{\text{Claim 6}}{=} x'^2y = xy \in F$.

Claim 8. *For every finite subset $\{x_1, \dots, x_t\} \in F$ the group $\overline{\langle x_1, \dots, x_t \rangle} \leq G$ is diagonalizable, i.e., is contained in a maximal torus of G . Moreover, there exists $m \in \mathbb{N}$ such that $\overline{\langle x_1, \dots, x_t \rangle} \cong (\mathbb{R}_{>0}^m, \cdot) \cong (\mathbb{R}^m, +)$.*

Since the weak flat F is closed, one has $\overline{\langle x_1, \dots, x_t \rangle} \leq F$. By Corollary 5.10 and the fact that maximal tori of G are closed (see [HKM13, Corollary 7.17]), each of the subgroups $H_i := \overline{\langle x_i \rangle} \leq F$ is diagonalizable. Moreover, $H_i \cong (\mathbb{R}, +)$ by direct computation in any torus containing x_i (see also Proposition 2.19). The groups H_i commute with one another by Claim 7, whence [Cap09, Proposition 4.4] implies that $\overline{\langle x_1, \dots, x_t \rangle}$ normalizes a maximal torus T of G . Moreover, since $W = N_G(T)/T$ is discrete and $\overline{\langle x_1, \dots, x_t \rangle}$ is connected, one actually has $\overline{\langle x_1, \dots, x_t \rangle} \leq T$. Connectedness then additionally implies that $\overline{\langle x_1, \dots, x_t \rangle} \leq T^\circ = \tau(T) \cong (\mathbb{R}^n, +)$, where n is the rank of G . (Note that in \overline{G} one obtains a torus isomorphic to $(\mathbb{R}^{\mathrm{rk}(\mathbf{A})}, +)$ instead.) The final statement follows from the classification of closed connected subgroups of $(\mathbb{R}^n, +)$.

Claim 9. *F is contained in a standard flat.*

Let $m := \max \left\{ \dim_{\mathbb{R}} \left(\overline{\langle x_1, \dots, x_t \rangle} \right) \mid t \in \mathbb{N}, x_1, \dots, x_t \in F \right\} \leq n$, where n is the rank of G , and let $\{x_1, \dots, x_t\} \subset F$ such that $\dim_{\mathbb{R}} \left(\overline{\langle x_1, \dots, x_t \rangle} \right) = m$. Then $F = \overline{\langle x_1, \dots, x_t \rangle}$: indeed, otherwise there exists $x_{t+1} \in F \setminus \overline{\langle x_1, \dots, x_t \rangle}$ and $\dim_{\mathbb{R}} \left(\overline{\langle x_1, \dots, x_t, x_{t+1} \rangle} \right) = m + 1$, a contradiction.

The proof for \overline{G} is essentially the same. □

Corollary 5.16. (i) G acts strongly transitively on G/K .

(ii) G and \overline{G} and $\text{Ad}(G)$ act strongly transitively on $\overline{G}/\overline{K}$.

(iii) Maximal flats in $\overline{G}/\overline{K}$ lift uniquely to maximal flats in G/K .

Proof. In view of Theorem 5.15 this follows from Corollary 5.5 and Proposition 2.26. Note that $\text{Ad}(G)$ indeed acts on $\overline{G}/\overline{K}$ by Lemma 4.5. \square

We have established Theorem 1.7.

6 Local and global automorphisms of Kac–Moody symmetric spaces

We keep the notation of the previous section, i.e. G denotes a simply connected centered split real Kac–Moody group satisfying the assumptions of Convention 4.2 on page 33, the group \overline{G} denotes its semisimple adjoint quotient, and $\text{Ad}(G)$ its adjoint quotient. Moreover, $\Delta = \Delta^- \sqcup \Delta^+$ denotes the common twin building associated to the canonical BN pairs of these groups.

6A Automorphisms of Kac–Moody groups

The abstract automorphisms of the groups G , \overline{G} and $\text{Ad}(G)$ have been classified in [Cap09]. Since \mathbb{R} does not admit any non-trivial field automorphism, this classification can be stated as follows.

Theorem 6.1 (Caprace [Cap09, Theorem 4.2]). *Let $\mathcal{G} \in \{G, \overline{G}, \text{Ad}(G)\}$. Then every automorphism of \mathcal{G} can be written as a product of an inner automorphism of \mathcal{G} , a diagram automorphism, a diagonal automorphism and a power of the Cartan–Chevalley involution θ .* \square

This result has several immediate consequences. Firstly, every automorphism of G preserves the center and hence descends to an automorphism of \overline{G} and $\text{Ad}(G)$. This is obvious for inner automorphisms and diagram automorphisms. For the Cartan–Chevalley involution this follows from the description of the center of G in Section 3C and the explicit description of θ in Section 3F. Finally, every non-trivial diagonal automorphism acts non-trivially on some root group, which can be seen in any central quotient of G . One thus obtains homomorphisms

$$\text{Aut}(G) \rightarrow \text{Aut}(\overline{G}) \rightarrow \text{Aut}(\text{Ad}(G)). \quad (6.1)$$

Secondly, it follows from the concrete description of automorphisms in Theorem 6.1 that every automorphism of $\text{Ad}(G)$ or \overline{G} can be extended to G . That is, the homomorphisms in (6.1) are isomorphisms.

Thirdly, if B is a Borel subgroup of G , i.e., a conjugate of B^+ or B^- , then every inner automorphism of G certainly maps B to a Borel subgroup; the same holds for diagram and diagonal automorphisms. Also, the Cartan–Chevalley involution swaps B^+ and B^- and thus preserves Borel subgroups. Since Borel subgroups can be identified with chambers of Δ , one observes that every automorphism of G induces an automorphism of Δ (not necessarily preserving the two halves), i.e., one obtains a homomorphism $\text{Aut}(G) \rightarrow \text{Aut}(\Delta)$.

Remark 6.2. Recall from Definition 3.2 that the Dynkin diagram $\Gamma_{\mathbf{A}}$ arises from the Coxeter diagram of (W, S) by adding certain labels. In particular, the automorphism group $\text{Aut}(\Gamma_{\mathbf{A}})$ of the Dynkin diagram is a subgroup of the automorphism group $\text{Aut}(W, S)$ of the Coxeter diagram. Some of the results in the sequel will require that the inclusion $\text{Aut}(\Gamma_{\mathbf{A}}) \hookrightarrow \text{Aut}(W, S)$ is an isomorphism. This hypothesis is satisfied for instance if the Dynkin diagram is simply-laced.

Corollary 6.3. *The homomorphism $\text{Aut}(G) \rightarrow \text{Aut}(\Delta)$ is injective, and thus*

$$\text{Aut}(G) \cong \text{Aut}(\overline{G}) \cong \text{Aut}(\text{Ad}(G)) \hookrightarrow \text{Aut}(\Delta).$$

If G is two-spherical and if $\text{Aut}(\Gamma_{\mathbf{A}}) = \text{Aut}(W, S)$, then it is an isomorphism, and thus

$$\text{Aut}(G) \cong \text{Aut}(\overline{G}) \cong \text{Aut}(\text{Ad}(G)) \cong \text{Aut}(\Delta).$$

Furthermore, $\text{Out}(G) := \text{Aut}(G)/\text{Inn}(G)$ is finite. Moreover, the $\text{Aut}(G)$ -conjugacy class of θ in $\text{Aut}(G)$ coincides with the $\text{Ad}(G)$ -conjugacy class of θ in $\text{Aut}(G)$.

Proof. The homomorphism $\text{Aut}(\text{Ad}(G)) \rightarrow \text{Aut}(\Delta)$ is injective, since $\text{Ad}(G)$ acts faithfully on Δ . Therefore it follows from the discussion above that $\text{Aut}(G) \rightarrow \text{Aut}(\Delta)$ is also a monomorphism.

Now assume that G is two-spherical and that $\text{Aut}(\Gamma_{\mathbf{A}}) = \text{Aut}(W, S)$. To prove surjectivity, one needs to prove that any automorphism $\alpha \in \text{Aut}(\Delta)$ is induced by an automorphism of G .

Each automorphism α of Δ induces a well-defined permutation of the diagram of Δ , which necessarily has to be an automorphism of the underlying Coxeter diagram. Hence the automorphism α is the product of a type-preserving automorphism of Δ and a Coxeter diagram automorphism. If G and Δ admit the same diagram automorphisms, i.e., if the automorphisms of the Dynkin diagram equal the automorphisms of the Coxeter diagram, one may assume that α is type-preserving.

Let C_+ and $C_- = \theta(C_+)$ be opposite chambers of the twin building Δ . By the strongly transitive action of G on Δ (see [AB08, Lemma 6.70 and Theorem 8.9]) there exists an inner automorphism of G that maps the set $\{\alpha(C_+), \alpha(C_-)\}$ onto the set $\{C_+, C_-\}$. By composing α with this inner automorphism and, if necessary, the Cartan–Chevalley involution θ , one may actually assume that the type-preserving automorphism α fixes the chambers C_+ and C_- .

If the diagram is two-spherical, then the extension theorem by Mühlherr and Ronan [MR95, Theorem 1.2] (see also [AB08, Theorem 5.213]) implies that the type-preserving automorphism α is the unique extension that fixes C_- of its restrictions to the residues of rank two containing C_+ in the positive half Δ_+ of the twin building Δ .

By inspection, those local rank-two restrictions are all induced by automorphisms of the corresponding split real Lie groups of rank two that as a family together provide an automorphism of the amalgam $\mathcal{A}(\mathbf{A})$ of fundamental subgroups of rank two of G . This amalgam automorphism, again using two-sphericity, induces a unique automorphism of G by [HKM13, Theorem 7.22] (see Section 3B) whose image under the natural map is α .

Concerning the finiteness of $\text{Aut}(G)/\text{Inn}(G)$, one notes that the automorphism group of a finite graph is finite and that the group generated by the involution θ is finite. By Theorem 6.1 it therefore suffices to observe that there are only finitely many diagonal automorphisms modulo inner automorphisms. This is implied by the fact, which is also true for (rational points of) algebraic groups, that the index of the adjoint quotient $\text{Ad}(G)$ inside the adjoint split real Kac–Moody group of type \mathbf{A} is finite: Indeed, the exact sequence $0 \rightarrow \mathcal{F} \rightarrow \overline{\mathcal{T}}^{sc} \rightarrow \mathcal{T}^{ad} \rightarrow 0$ of torus schemes (where $\overline{\mathcal{T}}^{sc}$ is the simply connected torus of the semisimple adjoint quotient \overline{G} and \mathcal{T}^{ad} is the torus of the adjoint Kac–Moody group of type \mathbf{A} and \mathcal{F} is defined as the kernel of $\overline{\mathcal{T}}^{sc} \rightarrow \mathcal{T}^{ad}$) yields an exact sequence $0 \rightarrow \mathcal{F}(\mathbb{R}) \rightarrow \overline{\mathcal{T}}^{sc}(\mathbb{R}) \rightarrow \mathcal{T}^{ad}(\mathbb{R}) \rightarrow H_{\text{ét}}^1(\mathbb{R}, \mathcal{F}) \rightarrow 0$ of \mathbb{R} -points; since étale cohomology is finite over \mathbb{R} (see, e.g., [Mil80]), the claim follows.⁵

The final claim follows from the fact that diagonal automorphisms and diagram automorphisms commute with θ . \square

⁵A more down-to-earth proof based on the same argument is as follows: By [Cap09, Theorem 4.1 and 4.2] a diagonal automorphism of G is a product of finitely many pairwise commuting diagonal automorphisms of the fundamental rank one subgroups $G_i \cong (\text{P})\text{SL}_2(\mathbb{R})$. The index of $\text{PSL}_2(\mathbb{R})$ inside $\text{PGL}_2(\mathbb{R})$ is two and, hence, $\text{SL}_2(\mathbb{R})$ has finitely many diagonal automorphisms modulo inner automorphisms. One concludes from the description of the diagonal automorphisms of G that also G only has finitely many diagonal automorphisms modulo inner automorphisms.

6B Automorphisms of the main group

The next main goal is to describe the automorphism group of the reduced Kac–Moody symmetric space $\overline{\mathcal{X}} = \overline{G}/\overline{K}$. Recall from Proposition 4.8 that

$$\text{Trans}(\overline{\mathcal{X}}) = \text{Ad}(G) \quad \text{and} \quad G(\overline{\mathcal{X}}) = \text{Ad}(G) \rtimes \langle \overline{\theta} \rangle.$$

Also recall from Remark 2.5 that there exists an embedding

$$c : \text{Aut}(\overline{\mathcal{X}}) \rightarrow \text{Aut}(G(\overline{\mathcal{X}})) = \text{Aut}(\text{Ad}(G) \rtimes \langle \overline{\theta} \rangle), \quad \alpha \mapsto c_\alpha,$$

where $c_\alpha(g) := \alpha \circ g \circ \alpha^{-1}$. Thus in order to determine $\text{Aut}(\overline{\mathcal{X}})$ one first needs to determine $\text{Aut}(\text{Ad}(G) \rtimes \langle \overline{\theta} \rangle)$.

Proposition 6.4. *Every automorphism α of $\text{Ad}(G)$ extends uniquely to an automorphism $\overline{\alpha}$ of $\text{Ad}(G) \rtimes \langle \overline{\theta} \rangle$, and this yields an isomorphism*

$$e : \text{Aut}(\text{Ad}(G)) \xrightarrow{\cong} \text{Aut}(\text{Ad}(G) \rtimes \langle \overline{\theta} \rangle), \quad \alpha \mapsto \overline{\alpha}.$$

Proof. Let $\alpha \in \text{Aut}(\text{Ad}(G))$. By Corollary 6.3, the automorphism α can be considered as an automorphism $\tilde{\alpha}$ of the building Δ . Since the automorphism

$$\tilde{\alpha} \circ \overline{\theta} \circ \tilde{\alpha}^{-1} \in \text{Aut}(\Delta)$$

is induced by a $\text{Ad}(G)$ -conjugate of the Cartan–Chevalley involution (cf. Corollary 6.3), the automorphism $\tilde{\alpha}$ normalizes the subgroup

$$\text{Ad}(G) \rtimes \langle \overline{\theta} \rangle < \text{Aut}(\Delta),$$

and, thus, induces an automorphism of $\text{Ad}(G) \rtimes \langle \overline{\theta} \rangle$ extending α . Since every element of $\text{Ad}(G) \rtimes \langle \overline{\theta} \rangle$ is uniquely determined by its action on the building, this is the only way to extend α and so yields the desired homomorphism e . If $\beta \in \text{Aut}(\text{Ad}(G) \rtimes \langle \overline{\theta} \rangle)$, then β and $e(\beta|_{\text{Aut}(\text{Ad}(G))})$ are both extensions of $\beta|_{\text{Aut}(\text{Ad}(G))}$ and therefore coincide; note here that $\text{Ad}(G)$ equals the commutator subgroup of $\text{Ad}(G) \rtimes \langle \overline{\theta} \rangle$, in particular is a characteristic subgroup, and as such invariant under β . This shows that e is an isomorphism. \square

6C Global automorphisms of reduced Kac–Moody symmetric spaces

In this section we prove Theorem 1.9.

Theorem 6.5. *The automorphism group of the reduced Kac–Moody symmetric space $\overline{\mathcal{X}} = \overline{G}/\overline{K}$ satisfies*

$$\text{Aut}(\overline{\mathcal{X}}) \cong \text{Aut}(G) \cong \text{Aut}(\overline{G}) \cong \text{Aut}(\text{Ad}(G)).$$

Proof. By Remark 2.5 there exists an embedding

$$c : \text{Aut}(\overline{\mathcal{X}}) \rightarrow \text{Aut}(\text{Ad}(G) \rtimes \langle \overline{\theta} \rangle), \quad \alpha \mapsto c_\alpha,$$

and it remains to show that every $\alpha \in \text{Aut}(\text{Ad}(G) \rtimes \langle \overline{\theta} \rangle) = \text{Aut}(G)$ is an image under this map. Theorem 6.1 allows one to conduct a case-by-case analysis of the different types of automorphisms.

Firstly, let $f = c_h$ be an inner automorphism of G given by conjugation by $h \in G$. Then h acts on $\overline{\mathcal{X}}$ by an automorphism $\alpha : gK \mapsto hgK$ and then $c_\alpha = c_h = f$, showing that f is in the image of c .

Secondly, let f be an automorphism of the Dynkin diagram. Then f acts on the family of the standard rank one subgroups of G by permuting the indices and acting trivially on each

group. This action commutes with the action of the Cartan–Chevalley involution θ (which acts by transpose-inverse on each fundamental rank one group), and thus f descends to a permutation α of G/K . One then checks that α preserves the multiplication μ and, in fact, $f = c_\alpha$. That is, f is contained in the image of c .

Thirdly, let $f = \theta$ be the Cartan–Chevalley involution. Since $\theta = c_{s_{eK}}$, one concludes that f is in the image of c .

Finally, let f be a diagonal automorphism of G . On each of the standard rank one subgroups $G_\alpha \cong (\text{P})\text{SL}_2(\mathbb{R})$, the restriction of f is given by conjugation by a diagonal matrix. A quick computation shows that this conjugation acts trivially on the circle subgroup $\text{SO}(2) < \text{SL}_2(\mathbb{R})$. It thus follows that f acts trivially on the group K , since it is generated by these circle groups. Consequently, f descends to a permutation α of G/K and $f = c_\alpha$ is in the image of c .

In view of Theorem 6.1 this finishes the proof. \square

Remark 6.6. By Theorem 6.1 the group $\text{Aut}(G)$ contains a subgroup $\text{Aut}^+(G)$ of index two, which is generated by all inner automorphisms, diagram automorphisms and diagonal automorphisms such that $\text{Aut}(G) = \text{Aut}^+(G) \rtimes \langle \theta \rangle$. In the sequel denote by $\text{Aut}^+(\mathcal{X})$ the image of $\text{Aut}^+(G)$ under the isomorphism $\text{Aut}(G) \rightarrow \text{Aut}(\mathcal{X})$ from Theorem 6.5. Then

$$\text{Aut}(\overline{\mathcal{X}}) = \text{Aut}^+(\overline{\mathcal{X}}) \rtimes \langle s_o \rangle, \quad (6.2)$$

where $o \in \overline{\mathcal{X}}$ is an arbitrary basepoint. Note that $\text{Aut}^+(\overline{\mathcal{X}})$ contains the transvection group $\text{Ad}(G)$.

6D Local automorphisms and the Coxeter complex

The next step is to describe the local automorphisms of $\overline{\mathcal{X}}$. We remind the reader that for a pointed maximal flat (p, F) the set $F^{\text{sing}}(p)$ of singular points of F with respect to p and the local automorphism group $\text{Aut}(p, F)$ were defined in Definition 2.28. By strong transitivity, these notions do not depend on the choice of pointed maximal flat up to isomorphism, and we will work with the *standard pointed flat* (e, \overline{AK}) of the coset model.

By Proposition 5.1 a chart of the pointed flat \overline{AK} centered at e is given by

$$\varphi_e : \overline{\mathfrak{a}} \rightarrow \overline{AK}, \quad X \mapsto \exp(X)\overline{K}.$$

Thus if one defines

$$\overline{\mathfrak{a}}^{\text{sing}} := \varphi_e^{-1}(\overline{AK}^{\text{sing}}(e)) \quad \text{and} \quad \text{GL}(\overline{\mathfrak{a}}, \overline{\mathfrak{a}}^{\text{sing}}) := \{f \in \text{GL}(\overline{\mathfrak{a}}) \mid f(\overline{\mathfrak{a}}^{\text{sing}}) = \overline{\mathfrak{a}}^{\text{sing}}\},$$

then one obtains an isomorphism

$$\text{Aut}(e, \overline{AK}) \xrightarrow{\cong} \text{GL}(\overline{\mathfrak{a}}, \overline{\mathfrak{a}}^{\text{sing}}), \quad F \mapsto f := \varphi_e^{-1} \circ F \circ \varphi_e; \quad (6.3)$$

In order to describe the local automorphism group more explicitly one needs a better understanding of the set $\overline{\mathfrak{a}}^{\text{sing}}$. By strong transitivity of \overline{G} (see Corollary 5.16), any two maximal flats through e in $\overline{G}/\overline{K}$ are \overline{K} -conjugate, i.e.

$$\overline{\mathfrak{a}}^{\text{sing}} = \bigcup_{\{k \in K \mid \text{Ad}(k)\mathfrak{a}_{\mathbb{R}} \neq \mathfrak{a}_{\mathbb{R}}\}} \overline{\mathfrak{a}} \cap \text{Ad}(k)\overline{\mathfrak{a}}, \quad (6.4)$$

The results from the appendix allow one to describe this set in a more combinatorial way. Recall from Definition A.7 the definition of the Kac–Moody representations $\rho_{KM} : W \rightarrow \text{GL}(\mathfrak{a})$ and the reduced Kac–Moody representation $\overline{\rho}_{KM} : W \rightarrow \text{GL}(\overline{\mathfrak{a}})$ of the Weyl group. As \mathbf{A} is assumed to be non-affine, both of these representations are faithful by Corollary A.11, and reflections in W act as linear reflections under these representations.

Given a real root $\alpha \in \Phi$ with associated root reflection $\check{r}_\alpha \in W$ we denote by $H_\alpha := \text{Fix}(\rho_{KM}(\check{r}_\alpha)) < \mathfrak{a}$ and $\overline{H}_\alpha := \text{Fix}(\overline{\rho}_{KM}(\check{r}_\alpha)) < \overline{\mathfrak{a}}$ the corresponding reflection hyperplanes in \mathfrak{a} and $\overline{\mathfrak{a}}$ respectively (cf. Definition A.8). Recall from (A.10) on page 70 that the reflections $\rho_{KM}(\check{r}_\alpha)$ and $\overline{\rho}_{KM}(\check{r}_\alpha)$ are orthogonal with respect to suitable choices of invariant bilinear forms on \mathfrak{a} and $\overline{\mathfrak{a}}$. Since the invariant form on $\overline{\mathfrak{a}}$ is non-degenerate (cf. the proof of Proposition A.10), the map $\overline{\rho}_{KM}(\check{r}_\alpha)$ is in fact the unique orthogonal reflection at the hyperplane \overline{H}_α . This implies in particular that the map $\alpha \mapsto \overline{H}_\alpha$ defines a one-to-one correspondence between positive real roots $\alpha \in \Phi^+$ and reflection hyperplanes \overline{H}_α .

Proposition 6.7. *Under the chart φ_e the singular set of the pointed maximal flat (e, \overline{A}) in $\overline{G}/\overline{K}$ corresponds to the union of the reflection hyperplanes of root reflections under $\hat{\rho}_{KM}$, i.e.*

$$\varphi_e^{-1}(\overline{A}^{\text{sing}}(e)) = \overline{\mathfrak{a}}^{\text{sing}} = \bigcup_{\alpha \in \Phi^+} \overline{H}_\alpha.$$

In particular,

$$\text{Aut}(e, \overline{A}) \cong \text{GL}(\overline{\mathfrak{a}}, \overline{\mathfrak{a}}^{\text{sing}}) = \left\{ f \in \text{GL}(\overline{\mathfrak{a}}) \mid f \left(\bigcup_{\alpha \in \Phi^+} \overline{H}_\alpha \right) = \bigcup_{\alpha \in \Phi^+} \overline{H}_\alpha \right\}.$$

Proof. Assume first that $X \in \overline{\mathfrak{a}}^{\text{sing}}$. By (6.4) there exists $k \in K$ with $\text{Ad}(k)\overline{\mathfrak{a}} \neq \overline{\mathfrak{a}}$ such that $X \in \overline{\mathfrak{a}} \cap \text{Ad}(k)\overline{\mathfrak{a}}$. Recall that \overline{T} is the unique maximal torus of \overline{G} such that $\overline{A} = \overline{T} \cap \overline{\tau}(\overline{G})$. Consequently, \overline{T}^k is the unique maximal torus of \overline{G} such that $\overline{A}^k = \overline{T}^k \cap \overline{\tau}(\overline{G})$. Assuming $X \neq 0$, one obtains a non-trivial intersection $H := \overline{T} \cap \overline{T}^k \ni \exp(X)$.

As in [Cap09, Proposition 4.6] let

$$\Phi^H = \{\alpha \in \Phi \mid [U_\alpha, H] = 1\} \quad \text{and} \quad \overline{G}^H = \overline{T} \cdot \langle U_\alpha \mid \alpha \in \Phi^H \rangle.$$

Since H is contained in the distinct tori \overline{T} and \overline{T}^k , it is not regular in the sense of [Cap09, Section 4.2.3], i.e., H fixes more than a single twin apartment of the twin building Δ . Hence [Cap09, Proposition 4.6(i)(ii)] imply that $(\overline{G}^H, (U_\alpha)_{\alpha \in \Phi^H})$ is a locally \mathbb{R} -split twin root datum with Weyl group $W^H = \langle s_\alpha \mid \alpha \in \Phi^H \rangle$ and maximal torus \overline{T} . Also \overline{T}^k is a maximal torus of \overline{G}^H by [Cap09, Proposition 4.6(v)], and \overline{G}^H centralizes H . Since \overline{G}^H acts transitively on twin apartments of the twin building associated with the twin root datum $(\overline{G}^H, (U_\alpha)_{\alpha \in \Phi^H})$ and these correspond to maximal tori in \overline{G}^H (see e.g. [AB08, Corollary 8.78]), one deduces that \overline{T} and \overline{T}^k are conjugate in \overline{G}^H .

Next observe that H is θ -invariant as \overline{T} and \overline{T}^k are. It then follows that for each $\alpha \in \Phi^H$ one has $-\alpha \in \Phi^H$, because

$$[U_\alpha, H] = 1 \quad \Longleftrightarrow \quad [U_{-\alpha}, H] = [\theta(U_\alpha), \theta(H)] = 1.$$

Therefore θ leaves $\langle U_\alpha, U_{-\alpha} \rangle$, $\alpha \in \Phi^H$ invariant and acts as an automorphism on \overline{G}^H . Consequently the group \overline{G}^H admits an Iwasawa decomposition $\overline{G}^H = \overline{K}^H \overline{A} \overline{U}^H$, where $\overline{K}^H \leq \overline{K} \cap \overline{G}^H$ and $\overline{U}^H \leq \overline{U}_+ \cap \overline{G}^H$.

The groups \overline{T} and \overline{T}^k are conjugate by an element $k^H \in [\overline{K}^H, \overline{K}^H]$: Indeed, since the group \overline{G}^H acts strongly transitively on the associated twin building, the θ -stable tori \overline{T} and \overline{T}^k are actually conjugate by an element k^H in \overline{K}^H . In fact, k^H can be chosen in $[\overline{K}^H, \overline{K}^H]$, since every element in \overline{K}^H is a product of such an element and an element centralizing H .

Now $[K^H, K^H]$ is generated by the family $(\overline{K} \cap \langle U_\alpha, U_{-\alpha} \rangle)_{\alpha \in \Phi^H}$ and, thus,

$$k^H = \prod_{i=1}^t k_i$$

for suitable $\beta_i \in \Pi^H$ and $k_i \in \overline{K} \cap \langle U_{\beta_i}, U_{-\beta_i} \rangle$. Not all elements k_i can normalize \overline{T} , for otherwise $\overline{T}^k = \overline{T}$. Pick $i \in \{1, \dots, t\}$ such that $\overline{T}^{k_i} \neq \overline{T}$. Then $H \leq \overline{T}^{k_i} \cap \overline{T}$, as $k_i \in \overline{G}^H$. Furthermore, $\overline{T}^{k_i} \cap \overline{T}$ has corank 1 in \overline{T} , because $\beta_i \in \Pi^H \subset \Phi^H \subset \Phi$ and $k_i \in \langle U_{\beta_i}, U_{-\beta_i} \rangle$. Now $\tilde{s}_{\beta_i} \in N_{\overline{K}^H}(A) \leq N_{\overline{K}}(\overline{A}) \leq N_{\overline{G}}(\overline{T})$ (cf. Section 3D and Lemma 3.21) fixes the intersection $\overline{T}^{k_i} \cap \overline{T}$ and, as it has corank one in \overline{T} , this intersection must be the exponential of the reflection hyperplane of H_{β_i} .

This shows that $X \in H_{\beta_i}$ and, since X was arbitrary, one obtains $\overline{\mathfrak{a}}^{\text{sing}} \subset \bigcup_{\alpha \in \Phi} H_{\alpha}$. Conversely, if $X \in H_{\alpha}$, then $\exp(X) \in \overline{A} \cap \overline{A}^k$, where $k \in K \cap \langle U_{\alpha}, U_{-\alpha} \rangle$ is any element not normalizing \overline{T} . \square

Note in passing that the same result also holds for the non-reduced Kac–Moody symmetric space.

Corollary 6.8. *Under the chart $\varphi_e : \mathfrak{a} \rightarrow AK$, $X \mapsto \exp(X)K$ the singular set of the pointed maximal flat (e, A) in G/K is given by*

$$\varphi_e^{-1}(A^{\text{sing}}(e)) = \mathfrak{a}^{\text{sing}} = \bigcup_{\alpha \in \Phi^+} H_{\alpha}.$$

Proof. This follows from Corollary 5.16 and Proposition 6.7; see also Proposition A.10. \square

Returning to the study of the reduced symmetric space, one also concludes from Proposition 6.7 that the subset $\overline{\mathfrak{a}}^{\text{sing}} \subset \overline{\mathfrak{a}}$ is precisely the hyperplane arrangement which is denoted by the same symbol $\overline{\mathfrak{a}}^{\text{sing}}$ in the appendix. In particular, Proposition A.17 applies. In view of our assumption that \mathbf{A} is non-spherical and non-affine, one deduces with Remark A.18 that the canonical linear realization (as defined in Subsection AG)

$$\overline{\rho} : \text{Aut}(\Sigma) \times \mathbb{Z}/2\mathbb{Z} \rightarrow \text{GL}(\overline{\mathfrak{a}}, \overline{\mathfrak{a}}^{\text{sing}}) \quad (6.5)$$

is an isomorphism. Here, $\Sigma = \Sigma(W, S)$ denotes the Coxeter complex of the Coxeter system (W, S) associated to \mathbf{A} , $\overline{\rho}(\text{Aut}(\Sigma))$ acts on $\overline{\mathfrak{a}}$ preserving the Tits cone $\mathcal{C} \subset \overline{\mathfrak{a}}$ (see Definition A.13), and the generator of $\mathbb{Z}/2\mathbb{Z}$ acts by $-\text{Id}_{\overline{\mathfrak{a}}}$ and thus interchanges the Tits cone and its negative.

Altogether one obtains the following:

Corollary 6.9. *For every pointed flat (p, F) in $\overline{\mathcal{X}}$ the local automorphism group $\text{Aut}(p, F)$ is isomorphic to $\text{Aut}(\Sigma) \times \mathbb{Z}/2\mathbb{Z}$. More precisely, if $\varphi : \overline{\mathfrak{a}} \rightarrow F$ is a chart centered at p , then the map*

$$\text{Aut}(e, \overline{A}) \rightarrow \text{Aut}(\Sigma) \times \mathbb{Z}/2\mathbb{Z}, \quad \alpha \mapsto \overline{\rho}^{-1}(\varphi^{-1} \circ \alpha \circ \varphi).$$

with $\overline{\rho}$ as in (6.5) is an isomorphism. \square

6E Local vs. global automorphisms

By Corollary 5.16 the group G and hence the full automorphism group $\text{Aut}(\mathcal{X})$ act strongly transitively on \mathcal{X} . In particular, the corresponding Weyl groups $W(\text{Aut}(\overline{\mathcal{X}}) \curvearrowright \overline{\mathcal{X}})$ and $W(G \curvearrowright \overline{\mathcal{X}})$ and local actions are well-defined (see Definition 2.30).

As before denote by $\Sigma = \Sigma(W, S)$ the Coxeter complex of the Coxeter system (W, S) underlying \mathbf{A} . We recall from Lemma A.16 that the automorphism group $\text{Aut}(\Sigma)$ splits as the semidirect product $\text{Aut}(\Sigma) = W \rtimes \text{Aut}(W, S)$, where $\text{Aut}(W, S)$ denotes the group of automorphisms of the Coxeter diagram. Also recall from Remark 6.2 that $\text{Aut}(\Gamma_{\mathbf{A}}) < \text{Aut}(W, S)$ and that equality holds for instance if the Dynkin diagram is simply laced.

Theorem 6.10. *Let (p, F) be a pointed flat in $\overline{\mathcal{X}}$, and identify $\text{Aut}(p, F) \cong \text{Aut}(\Sigma) \times \mathbb{Z}/2\mathbb{Z}$ as in Corollary 6.9. Then the image of the local action*

$$W(\text{Aut}(\overline{\mathcal{X}}) \curvearrowright \overline{\mathcal{X}}) \rightarrow \text{Aut}(p, F)$$

is given by the subgroup

$$(W \rtimes \text{Aut}(\Gamma_{\mathbf{A}})) \times \mathbb{Z}/2\mathbb{Z} < (W \rtimes \text{Aut}(W, S)) \times \mathbb{Z}/2\mathbb{Z} = \text{Aut}(\Sigma) \times \mathbb{Z}/2\mathbb{Z} \cong \text{Aut}(p, F),$$

and the image of the local action $W(\text{Aut}^+(\overline{\mathcal{X}}) \curvearrowright \overline{\mathcal{X}}) \rightarrow \text{Aut}(p, F)$ is given by the index 2 subgroup $(W \rtimes \text{Aut}(\Gamma_{\mathbf{A}}))$. In particular, any local automorphism of $\overline{\mathcal{X}}$ extends to a global automorphism if and only if the automorphism group of the Coxeter diagram equals the automorphism group of the Dynkin diagram.

Proof. One may assume that $(p, F) = (e, \overline{A})$ and identify $\text{Aut}(e, \overline{A})$ with $\text{Aut}(\Sigma) \times \mathbb{Z}/2\mathbb{Z}$ by the explicit isomorphism in Corollary 6.9. Under this identification, the generator of $\mathbb{Z}/2\mathbb{Z}$ is induced by the Cartan–Chevalley involution θ , and every element of W is induced by an element of K (see Subsection 3D and Proposition 3.14). Finally, the diagram automorphism in $\text{Aut}(G) \cong \text{Aut}(\overline{\mathcal{X}})$ induce the automorphisms in the subgroup $\text{Aut}(\Gamma_{\mathbf{A}}) < \text{Aut}(W, S)$. \square

The group G admits two different Weyl groups: The (algebraic) Weyl group W and the (geometric) Weyl group $W(G \curvearrowright \overline{\mathcal{X}})$ arising from the action of G on the reduced symmetric space $\overline{\mathcal{X}}$. As is the case for finite-dimensional semisimple Lie group, the algebraic and the geometric Weyl group coincide.

Theorem 6.11. *The local action $W(G \curvearrowright \overline{\mathcal{X}}) \rightarrow \text{Aut}(\Sigma)$ induces an isomorphism*

$$W(G \curvearrowright \overline{\mathcal{X}}) \cong W.$$

Proof. By Theorem 6.10 and its proof there are embeddings

$$W(G \curvearrowright \overline{\mathcal{X}}) \hookrightarrow W(\text{Aut}(\overline{\mathcal{X}}) \curvearrowright \overline{\mathcal{X}}) \cong \text{Aut}(\Sigma) \times \mathbb{Z}/2\mathbb{Z} = (W \rtimes \text{Aut}(W, S)) \times \mathbb{Z}/2\mathbb{Z}.$$

As discussed in the proof of Theorem 6.10, any element in W is induced by an element of $\text{Stab}_G(p, F) < K$. Since W is assumed to be non-spherical, a diagram automorphism in $\text{Aut}(W, S)$ can, if at all, only be induced by a Dynkin diagram automorphism of G ; the generator of $\mathbb{Z}/2\mathbb{Z}$ corresponds to the Cartan–Chevalley involution θ . The claim follows. \square

We have shown Theorem 1.11 and Corollary 1.12.

7 Causal structures and the causal boundary

We keep the notation of the previous section, i.e., G denotes a simply connected centered split real Kac–Moody group satisfying the assumptions of Convention 4.2 on page 33, the group \overline{G} denotes its semisimple adjoint quotient, and $\text{Ad}(G)$ its adjoint quotient. Moreover, $\Delta = \Delta^- \sqcup \Delta^+$ denotes the common twin building associated to the canonical BN pairs of these groups. We consider the reduced Kac–Moody symmetric space $\overline{\mathcal{X}} = \overline{G}/\overline{K}$.

7A Invariant causal structures

The goal of this subsection is to introduce an $\text{Aut}(\overline{\mathcal{X}})$ -invariant field of double cones in $\overline{\mathcal{X}}$. Our starting point is the observation that the vector space $\overline{\mathfrak{a}}$ contains a canonical cone $\mathcal{C} \subset \overline{\mathfrak{a}}$ with open interior \mathcal{C}° and tip 0, which is called the *Tits cone*, see Definition A.13 in the appendix. Under our

standing assumptions that \mathbf{A} be irreducible, non-spherical and non-affine this cone is *pointed* in the sense that

$$\mathcal{C} \cap (-\mathcal{C}) = \{0\}.$$

Refer to the union $\mathcal{C}^o \cup (-\mathcal{C}^o)$ as the *open Tits double cone* in $\bar{\mathbf{a}}$. Denote by $\bar{A}_{\pm} := \exp(\pm \mathcal{C}^o)$ the corresponding subsemigroups of \bar{A} and refer to $\bar{A}_+ \cup \bar{A}_-$ as the *canonical double cone* in \bar{A} . Let now F be an arbitrary flat through e in the group model of $\bar{\mathcal{X}}$. By strong transitivity, there exists $k \in \text{Aut}(\bar{\mathcal{X}})_e$ such that $k \cdot \bar{A} = F$. Moreover, the subset $\bar{\mathcal{C}}_e^{+, -}(F) := k \cdot (\bar{A}_+ \cup \bar{A}_-) \subset F$ is independent of the choice of k . Indeed, if k' is a different choice, then $k^{-1}k'$ acts on $\bar{\mathbf{a}}$ by a local automorphism, and by Theorem 6.10 any such automorphism leaves the canonical double cone invariant. Define

$$\bar{\mathcal{C}}_e^{+, -} := \bigcup \bar{\mathcal{C}}_e^{+, -}(F),$$

where the union is taken over all flats containing the basepoint e . Note that $\bar{\mathcal{C}}_e^{+, -} \cap F = \bar{\mathcal{C}}_e^{+, -}(F)$, and thus $\bar{\mathcal{C}}_e^{+, -}$ intersects each flat in a double cone, whose two halves do not intersect. By abuse of language, also call $\bar{\mathcal{C}}_e^{+, -}$ a double cone.

By construction, the double cone $\bar{\mathcal{C}}_e^{+, -}$ is invariant under all automorphisms in $\text{Aut}(\bar{\mathcal{X}})_e$. In particular, if $x \in \bar{\mathcal{X}}$ and if $\alpha \in \text{Aut}(\bar{\mathcal{X}})$ maps e to x , then

$$\bar{\mathcal{C}}_x^{+, -} := \alpha(\text{Aut}(\bar{\mathcal{X}})_e)$$

is independent of the choice of α . Moreover, if $\varphi : \bar{\mathbf{a}} \rightarrow F$ is any chart centered at x , then

$$\bar{\mathcal{C}}_x^{+, -}[F] := \bar{\mathcal{C}}_x^{+, -} \cap F = \varphi(\mathcal{C}^o \cup (-\mathcal{C}^o)).$$

Note also that by construction the family $(\bar{\mathcal{C}}_x^{+, -})_{x \in \bar{\mathcal{X}}}$ of double cones is $\text{Aut}(\bar{\mathcal{X}})$ -invariant in the sense that

$$\alpha(\bar{\mathcal{C}}_x^{+, -}) = \bar{\mathcal{C}}_{\alpha(x)}^{+, -} \quad (\alpha \in \text{Aut}(\bar{\mathcal{X}}), x \in \bar{\mathcal{X}}).$$

Refer to $(\bar{\mathcal{C}}_x^{+, -})_{x \in \bar{\mathcal{X}}}$ as the *canonical double cone field* on $\bar{\mathcal{X}}$.

If $\varphi, \varphi' : (0, \bar{\mathbf{a}}) \rightarrow (p, F)$ are charts, then $\varphi^{-1} \circ \varphi'$ is a linear map preserving the decomposition $\bar{\mathbf{a}} = \bar{\mathbf{a}}^{\text{reg}} \sqcup \bar{\mathbf{a}}^{\text{sing}}$ as well as the open double Tits cone $\mathcal{C}^o \cup (-\mathcal{C}^o) \subset \bar{\mathbf{a}}$. There are thus two possibilities: Either $\varphi^{-1} \circ \varphi'$ preserves the open Tits cone or it maps the open Tits cone to its negative.

Definition 7.1. Two charts $\varphi, \varphi' : (0, \bar{\mathbf{a}}) \rightarrow (p, F)$ of F centered at the same point p are called *causally equivalent* if $\varphi^{-1} \circ \varphi'$ preserves the open Tits cone. A *causal orientation* of $\bar{\mathcal{X}}$ is a choice of one of the two causal equivalence classes of charts for every maximal pointed flat (p, F) .

If a group H acts by automorphisms on $\bar{\mathcal{X}}$, then a causal orientation is called *H-invariant* if for every $h \in H$ and every chart φ in the chosen causal equivalence class, also the chart $h \circ \varphi$ is in the chosen equivalence class.

Proposition 7.2. *There exist exactly two $\text{Aut}^+(\bar{\mathcal{X}})$ -invariant causal orientations on $\bar{\mathcal{X}}$.*

Proof. Since $\text{Aut}^+(\bar{\mathcal{X}})$ acts strongly transitively on $\bar{\mathcal{X}}$ and since every pointed maximal flat admits only two causal equivalence classes, there are at most two G -invariant causal structures on $\bar{\mathcal{X}}$. By Theorem 6.10 one has $W(\text{Aut}(\bar{\mathcal{X}}) \curvearrowright \bar{\mathcal{X}}) \cong (W \rtimes \text{Aut}(\Gamma_{\mathbf{A}})) \times \mathbb{Z}/2\mathbb{Z}$, where the first factor acts on the Tits cone, and the second factor swaps the Tits cone and its negative. Moreover, $W(\text{Aut}^+(\bar{\mathcal{X}}) \curvearrowright \bar{\mathcal{X}})$ is given by the subgroup $(W \rtimes \text{Aut}(\Gamma_{\mathbf{A}})) \times \{e\}$. One thus obtains two distinct $\text{Aut}^+(\bar{\mathcal{X}})$ -invariant causal orientations, one for which the charts $\{\alpha \circ \exp \mid \alpha \in \text{Aut}^+(\bar{\mathcal{X}})\}$ are positive, and one for which the charts $\{-\alpha \circ \exp \mid \alpha \in \text{Aut}^+(\bar{\mathcal{X}})\}$ are positive. \square

Charts in the unique $\text{Aut}^+(\bar{\mathcal{X}})$ -invariant causal orientation containing \exp are called *positive charts*, charts in the unique $\text{Aut}^+(\bar{\mathcal{X}})$ -invariant causal orientation containing $-\exp$ *negative charts*. Given a pointed maximal flat (x, F) in $\bar{\mathcal{X}}$ and a positive chart $\varphi : \bar{\mathbf{a}} \rightarrow F$ centered at x define

$$\bar{\mathcal{C}}_x^+[F] := \varphi(\mathcal{C}^o) \quad \text{and} \quad \bar{\mathcal{C}}_x^-[F] := \varphi(-\mathcal{C}^o).$$

By definitions, these cones do not depend on the choice of positive chart, and if one defines

$$\overline{\mathcal{C}}_x^\pm := \bigcup_{F \ni x} \overline{\mathcal{C}}_x^\pm[F],$$

then $\overline{\mathcal{C}}_x^{+,-} = \overline{\mathcal{C}}_x^+ \cup \overline{\mathcal{C}}_x^-$. This decomposes the canonical double cone field on $\overline{\mathcal{X}}$ into two cone fields.

Definition 7.3. The cone field $(\overline{\mathcal{C}}_x^+)_{x \in \overline{\mathcal{X}}}$ is called the *positive causal structure* on $\overline{\mathcal{X}}$, and the cone field $(\overline{\mathcal{C}}_x^-)_{x \in \overline{\mathcal{X}}}$ is called the *negative causal structure* on $\overline{\mathcal{X}}$.

Note that the positive and negative causal structure are invariant under $\text{Aut}^+(\overline{\mathcal{X}})$, and in particular G -invariant. At this point we have established Proposition 1.13.

Remark 7.4. In Lorentzian geometry, invariant causal structures arise naturally. Namely, if $(g_x)_{x \in X}$ is a Lorentzian metric on a manifold X , then the associated field of light cones $(\mathcal{C}_x \subset T_x X)_{x \in X}$ is invariant under all Lorentzian automorphisms. In our setting, there is always an invariant bilinear form on $\overline{\mathfrak{a}}$, since \mathbf{A} is assumed to be symmetrizable. However, this bilinear form need not be Lorentzian, and even if it is Lorentzian it may happen that the Tits cone is not contained in the light cone of the invariant Lorentzian form (see e.g. [FKN12]). We emphasize that our G -invariant causal structures are modelled on the Tits cone, rather than the light cone of a bilinear form, hence our geometry here is *causal* rather than *Lorentzian*. This being said, in certain hyperbolic examples, including E_{10} , the interiors of the Tits cone and the light cone coincide according to [FKN12], [CFF16]; hence in these cases our results do admit a Lorentzian interpretation. In these examples our construction of causal boundaries below is a global version of the lightcone embedding provided in [CFF16].

7B Causal geodesic rays and the municipality

The positive causal structure gives rise to a notion of causal curves in the following standard way:

Definition 7.5. Let I be a non-trivial closed or half-open interval. A continuous map $\gamma : I \rightarrow \overline{\mathcal{X}}$ is called a *causal curve* if for every $t \in I$ there exists $\varepsilon > 0$

$$\gamma((t, t + \varepsilon)) \subset \overline{\mathcal{C}}_{\gamma(t)}^+.$$

If instead for every $t \in I$ there exists $\varepsilon > 0$

$$\gamma((t, t + \varepsilon)) \subset \overline{\mathcal{C}}_{\gamma(t)}^-,$$

then γ is called an *anti-causal curve*.

A (anti-)causal curve, which is also a geodesic ray, respectively a geodesic segment, will be called a *(anti-)causal ray*, respectively *(anti-)causal segment*.

Lemma 7.6. Let $r : [0, \infty) \rightarrow \overline{\mathcal{X}}$ be a geodesic ray, let $0 < S < T < \infty$ and let $\gamma : [S, T] \rightarrow \overline{\mathcal{X}}$ be the geodesic segment obtained by restricting r to $[S, T]$. Then the following are equivalent.

- (i) γ is a causal segment.
- (ii) $r(t) \in \overline{\mathcal{C}}_{r(0)}^+$ for some $t \in \mathbb{R}$.
- (iii) $r(t) \in \overline{\mathcal{C}}_{r(s)}^+$ for all $0 \leq s < t < \infty$.
- (iv) $\gamma(t) = r(t) \in \overline{\mathcal{C}}_{r(s)}^+$ for all $S \leq s < t \leq T$.

(v) r is a causal ray.

Proof. The implications (i) \Rightarrow (ii), (iii) \Rightarrow (iv) \Rightarrow (i) and (iii) \Rightarrow (v) \Rightarrow (i) are immediate from the definitions. To show the remaining implication (ii) \Rightarrow (iii) one may assume by strong transitivity, that r is contained in \overline{A} and emanates from e , i.e., $r(t) = \exp(tX)$ for some $X \in \overline{\mathfrak{a}}$. Under this assumption, (ii) amounts to $tX \in \mathcal{C}$ for some $t \in \mathbb{R}$. This implies that $(t-s)X \in \mathcal{C}$ for all $0 \leq s < t < \infty$, which is (iii). \square

In the sequel $\partial_\bullet \overline{\mathcal{X}}$ denotes the collection of all geodesic rays $r : [0, \infty) \rightarrow \overline{\mathcal{X}}$. Then $\partial_\bullet \overline{\mathcal{X}}$ fibers over $\overline{\mathcal{X}}$ by the map

$$\text{ev}_0 : \partial_\bullet \overline{\mathcal{X}} \rightarrow \overline{\mathcal{X}}, \quad r \mapsto r(0),$$

and we refer to the fiber $\partial_x \overline{\mathcal{X}} := \text{ev}_0^{-1}(x)$ over x as the *point horizon* of x . Given a flat F containing x we also denote by $\partial_\bullet \overline{\mathcal{X}}[F] \subset \partial_\bullet \overline{\mathcal{X}}$ the subset of rays emanating from x and contained in F . The action of the automorphism group $\text{Aut}(\overline{\mathcal{X}})$ preserves geodesic rays and thus induces an action on $\partial_\bullet \overline{\mathcal{X}}$, for which the projection ev_0 is equivariant. In particular, for every $x \in \overline{\mathcal{X}}$ the point stabilizer $\text{Aut}(\overline{\mathcal{X}})_x$ acts on $\partial_x \overline{\mathcal{X}}$, and $\text{Aut}(p, F)$ acts on $\partial_\bullet \overline{\mathcal{X}}[F]$.

To explicitly parametrize geodesic rays in $\overline{\mathcal{X}}$, consider again the standard pointed maximal flat (e, \overline{A}) in the group model of $\overline{\mathcal{X}}$. Then the geodesic rays contained in \overline{A} and emanating from e are given by $r_{e,X}(t) := \exp(tX)$, where X runs through the Lie algebra $\overline{\mathfrak{a}}$. Since $\overline{\mathcal{X}}$ is strongly transitive, every geodesic ray in $\overline{\mathcal{X}}$ is of the form $r_{g,X}(t) := g \cdot \exp(tX)$ for some $g \in \text{Aut}(\overline{\mathcal{X}})$ and $X \in \overline{\mathfrak{a}}$. One thus obtains a surjective map

$$\text{Aut}(\overline{\mathcal{X}}) \times \overline{\mathfrak{a}} \rightarrow \partial_\bullet \overline{\mathcal{X}}, \quad (g, X) \mapsto r_{g,X}.$$

Note that this map is not injective, i.e. the ray $r_{g,X}$ does not determine the parameters g and X .

Definition 7.7. A geodesic ray $r : [0, \infty) \rightarrow \overline{\mathcal{X}}$ is called *regular* if it is contained in a unique maximal flat of $\overline{\mathcal{X}}$ and *singular* otherwise.

Note that by Lemma 2.15 these notions are invariant under automorphisms of $\overline{\mathcal{X}}$. Recall the notation $\overline{\mathfrak{a}}^{\text{sing}} := \log(\overline{A}^{\text{sing}}(e))$ for the logarithm of the singular set of (e, \overline{A}) from Subsection 6D; denote by $\overline{\mathfrak{a}}^{\text{reg}} := \overline{\mathfrak{a}} \setminus \overline{\mathfrak{a}}^{\text{sing}}$ its complement. In terms of the parametrization above, regular and singular geodesic rays can be characterized as follows.

Lemma 7.8. The geodesic ray $r_{g,X}$ is singular if $X \in \overline{\mathfrak{a}}^{\text{sing}}$ and regular if $X \in \overline{\mathfrak{a}}^{\text{reg}}$.

Proof. By invariance of regular/singular rays under automorphisms it suffices to show this for $g = e$. It therefore remains to prove that, if $X \in \overline{\mathfrak{a}}^{\text{reg}}$, then the whole open ray $\{tX \mid t \in (0, \infty)\}$ is contained in $\overline{\mathfrak{a}}^{\text{reg}}$. This, however, follows from the fact that $\overline{\mathfrak{a}}^{\text{sing}}$ is a hyperplane arrangement by Proposition 6.7. \square

Definition 7.9. The subset $\Delta_\bullet \subset \partial_\bullet \overline{\mathcal{X}}$ consisting of all causal and anti-causal rays is called the *municipality* of $\overline{\mathcal{X}}$.

The terminology refers to the fact, to be proved in Proposition 7.17 below, that the fibers of Δ_\bullet are geometric realizations of the twin building of G . By construction, $\Delta_\bullet \subset \partial_\bullet \overline{\mathcal{X}}$ is $\text{Aut}(\overline{\mathcal{X}})$ -invariant, and if one denotes by $\Delta_\bullet^\pm \subset \Delta_\bullet$ the collections of causal/anti-causal rays, then these are invariant under $\text{Aut}^+(\overline{\mathcal{X}})$. Also note that one can characterize causal/anti-causal rays in terms of the standard parametrization as follows.

Proposition 7.10. The ray $r_{g,X}$ is contained in Δ_\bullet if and only if $X \in \mathcal{C}^o \cup -\mathcal{C}^o$. \square

Denote by

$$\Delta_\bullet^{\text{reg}} = \{r_{g,X} \in \Delta_\bullet \mid X \in \overline{\mathfrak{a}}^{\text{reg}}\}, \text{ respectively } \Delta_\bullet^{\text{sing}} = \{r_{g,X} \in \Delta_\bullet \mid X \in \overline{\mathfrak{a}}^{\text{sing}}\},$$

the subsets of regular, respectively singular rays in the municipality. Furthermore, given $x \in \overline{\mathcal{X}}$, denote by Δ_x , Δ_x^{reg} and Δ_x^{sing} the corresponding fibers over x .

Since the notion of a municipality ray is invariant under automorphisms, the subset $\Delta_\bullet \subset \partial_\bullet \overline{\mathcal{X}}$ is $\text{Aut}(\overline{\mathcal{X}})$ -invariant, and the induced $\text{Aut}(\overline{\mathcal{X}})$ -action preserves the decomposition $\Delta_\bullet = \Delta_\bullet^{\text{reg}} \sqcup \Delta_\bullet^{\text{sing}}$. Consequently, for every $x \in \overline{\mathcal{X}}$ the point stabilizer $\text{Aut}(\overline{\mathcal{X}})_x$ acts on Δ_x preserving the decomposition $\Delta_x = \Delta_x^{\text{reg}} \sqcup \Delta_x^{\text{sing}}$.

7C The simplicial structure of the municipality

The goal of this subsection is to define for every $x \in \overline{\mathcal{X}}$ a simplicial structure on the corresponding fiber Δ_x and to show that the resulting simplicial space is $\text{Aut}(\overline{\mathcal{X}})_x$ -equivariantly isomorphic to the geometric realization of the twin building of G .

Remark 7.11. We will use the following conventions and notations concerning simplicial complexes. Given an abstract simplicial complex (\mathcal{S}, \leq) denote by $|\mathcal{S}|$ the geometric realization of (\mathcal{S}, \leq) . A *simplicial structure* on a set X is a bijection $\varphi : |\mathcal{S}| \rightarrow X$ for some abstract simplicial complex (\mathcal{S}, \leq) . The pair (X, φ) is called a *simplicial space*; the images of simplices in $|\mathcal{S}|$ under φ are called *simplices* of X . Given two sets X_1, X_2 with simplicial structures $\varphi_i : |\mathcal{S}_i| \rightarrow X_i$, a bijection $X_1 \rightarrow X_2$ is called a *simplicial isomorphism* provided it maps simplices of simplices and on each simplex preserves barycentric coordinates. With this terminology, simplicial isomorphisms from X_1 to X_2 are in one-to-one correspondence with isomorphisms of the underlying abstract simplicial complexes.

For a fixed $x \in \overline{\mathcal{X}}$, one can construct a simplicial structure on Δ_x as follows. Given a maximal flat F containing x refer to the connected components of

$$\bigcup_{\gamma \in \Delta_x^{\text{reg}}} \gamma((0, \infty)) \cap F = F^{\text{reg}}(x) \cap (\mathcal{C} \cup -\mathcal{C})$$

as *(open) Weyl chambers* with tip x in F , and to their closures as *closed Weyl chambers*. Intersections of closed Weyl chambers will be referred to as *Weyl faces*.

Proposition 7.12. *Let (x, F) be a pointed maximal flat in $\overline{\mathcal{X}}$. If $\varphi : \overline{\mathfrak{a}} \rightarrow F$ is a chart centered at x , then the pre-images of closed Weyl chambers under φ are closed simplicial cones with tip 0 and non-empty interior. Any two such simplicial cones intersect along faces.*

Proof. If φ, φ' are two different charts of F centered at x , then they differ only by a linear automorphism of $\overline{\mathfrak{a}}$. Since the claim is invariant under such automorphisms, it suffices to establish it for one specific choice of φ . Now for a suitable choice of chart the pre-images of the Weyl chambers in \mathfrak{a} are precisely the open Tits chambers and their negatives in the sense of Definition A.13, and these have the desired properties. \square

Remark 7.13. Let $\mathcal{W}_x(F)$ be the collection of all Weyl faces with tip x in F different from $\{x\}$. We warn the reader that while every closed Weyl chamber in F with tip x is closed in F , their union need not be closed. In fact, under a suitable chart $\overline{\mathfrak{a}} \rightarrow F$ the union of these closed Weyl chambers corresponds to the double Tits cone in $\overline{\mathfrak{a}}$, which may fail to be closed.

As a consequence of Proposition 7.12, the set $\mathcal{W}_x(F)$ is an abstract simplicial complex with respect to inclusion. Since $\mathcal{C} \cap -\mathcal{C} = \{0\}$ and since $\{x\}$ is not contained in $\mathcal{W}_x(F)$, the simplicial complex $\mathcal{W}_x(F)$ (or equivalently, its realization) is not connected, but has two connected components, corresponding to causal and anti-causal rays respectively.

If one denotes by $\Delta_x(F) \subset \Delta_x$ the subset of rays contained in F , then one obtains an explicit identification between the set $\Delta_x(F)$ and the geometric realization $|\mathcal{W}_x(F)|$ of the simplicial complex $(\mathcal{W}_x(F), \subseteq)$ as follows. Fix a chart $\varphi : \overline{\mathfrak{a}} \rightarrow F$ centered at x and identify each closed Weyl chamber C with its pre-image $\varphi^{-1}(C) \subset \overline{\mathfrak{a}}$. The intersection of $\varphi^{-1}(C)$ with the unit sphere

$S_a \subset \bar{a}$ then defines a spherical simplex, and one obtain a continuous map $|\mathcal{W}_x(F)| \rightarrow S_a$ which identifies the realization of C with $\varphi^{-1}(C) \cap S_a \subset S_a$. On the other hand one can identify S_a with the set $\partial_x(F)$ of geodesic rays in F emanating from x by identifying $v \in S_a$ with the geodesic ray $\varphi(tv)$. Under this identification the image of the map $i_{x,F} : |\mathcal{W}_x(F)| \rightarrow S_a \cong \partial_x(F)$ is precisely given by $\Delta_x(F)$. In particular, one obtains a simplicial structure on $\Delta_x(F)$ by means of the bijection

$$i_{x,F} : |\mathcal{W}_x(F)| \rightarrow \Delta_x(F).$$

This simplicial structure does not depend on the choice of chart φ , and hence we refer to it as the *canonical simplicial structure* of $\Delta_x(F)$.

Note that each of the two connected components of $\mathcal{W}_x(F)$ is isomorphic as a simplicial complex to the Tits cone in \bar{a} , or equivalently, the Coxeter complex $\Sigma = \Sigma(W, S)$ of the Coxeter system (W, S) associated to the underlying generalized Cartan matrix \mathbf{A} . One concludes the following:

Corollary 7.14. *For every pointed flat (x, F) the set $\Delta_x(F)$ with its canonical simplicial structure is simplicially isomorphic to the geometric realization of a twin apartment in the twin building Δ of G . Under any such isomorphism the subsets $\Delta_x^\pm(F) \subset \Delta_x(F)$ correspond to the two halves of the twin apartment.* \square

Now consider two distinct flats F, F' through x , and denote by $W := F \cap F' \subset F$ their intersection. If one fixes a chart $\varphi : \bar{a} \rightarrow F$ centered at x , then W is contained in $F^{\text{sing}}(x)$, which under φ corresponds to a hyperplane arrangement in \bar{a} . If F' intersects F in codimension 1, then W is the image of a hyperplane in \bar{a} , and in general it will be an intersection of such hyperplanes corresponding to maximal flats intersecting F in codimension 1.

It follows that if one sets

$$\mathcal{W}_x(F, W) := \{C \in \mathcal{W}_x(F) \mid C \subset W\},$$

then $\mathcal{W}_x(F, W) \subset \mathcal{W}_x(F)$ is a simplicial subcomplex, and hence inherits the structure of a simplicial complex. Moreover, the simplicial complexes $\mathcal{W}_x(F, W)$ and $\mathcal{W}_x(F', W)$ coincide. Hence the union

$$\mathcal{W}_x := \bigcup_{F \ni x} \mathcal{W}_x(F)$$

is a simplicial complex.

Lemma 7.15. *The family of bijection $i_{x,F} : |\mathcal{W}_x(F)| \rightarrow \Delta_x(F)$ combine into an $\text{Aut}(\bar{\mathcal{X}})_x$ -equivariant bijection*

$$i_x : |\mathcal{W}_x| \rightarrow \Delta_x.$$

Proof. Firstly, the $i_{x,F}$ combine into i_x , since if F and F' intersect, then $i_{x,F}$ and $i_{x,F'}$ agree on $|\mathcal{W}_x(F, F \cap F')|$ and map it to $\Delta_x(F) \cap \Delta_x(F')$. Equivariance follows from the fact that the construction of $i_{x,F}$ is independent of the choice of chart together with the fact that every $g \in \text{Aut}(\bar{\mathcal{X}})_x$ maps charts to charts. \square

Definition 7.16. The simplicial structure on Δ_x given by the bijection i_x from Lemma 7.15 is called the *canonical simplicial structure* on Δ_x .

Note that the group $\text{Aut}(\bar{\mathcal{X}})_x$ acts by automorphisms on the simplicial complex $(\mathcal{W}_x, \subseteq)$, and thus $\text{Aut}(\bar{\mathcal{X}})_x$ acts on Δ_x by simplicial automorphisms. Via the canonical embedding $\text{Aut}(\bar{\mathcal{X}}) \hookrightarrow \text{Aut}(\Delta)$, it also acts on the twin building Δ associated with G , and hence by simplicial automorphisms on $|\Delta|$.

Proposition 7.17. *For every $x \in \bar{\mathcal{X}}$ there is an $\text{Aut}(\bar{\mathcal{X}})_x$ -equivariant isomorphism $\Delta \rightarrow \mathcal{W}_x$, which induces an $\text{Aut}(\bar{\mathcal{X}})_x$ -equivariant simplicial isomorphism $|\Delta| \rightarrow \Delta_x$.*

Proof. For every $x \in \overline{\mathcal{X}}$ there is an $\text{Aut}(\overline{\mathcal{X}})_x$ -equivariant bijection $\varphi_x : \mathcal{F}_x \rightarrow \mathcal{A}_x$ between the set \mathcal{F}_x of flats containing x and the set \mathcal{A}_x of twin apartments of Δ which are invariant under s_x : By transitivity of $\text{Aut}(\overline{\mathcal{X}})$ on $\overline{\mathcal{X}}$ it suffices to show this for $x = e$, the basepoint of the group model. Recall from (5.3) that maximal flats are in G -equivariant bijection with maximal tori of G , hence with twin apartments in Δ . Since the point reflection s_e is induced by θ , the flats through e correspond to θ -stable tori and thus to twin apartments which are invariant under s_e , and this correspondence is equivariant with respect to the point stabilizer K of e in G . To see that the correspondence is $\text{Aut}(\overline{\mathcal{X}})_e$ -equivariant one can argue as follows: By Theorem 6.5 one has $\text{Aut}(\overline{\mathcal{X}})_e \cong \text{Aut}(G)_e$ and every element of $\text{Aut}(G)_e$ is a product of an element of K with an automorphism which fixes both the flat \overline{A} through e and the corresponding twin apartment $\varphi_e(\overline{A})$ of Δ . It follows that the given bijection is not only K -equivariant, but moreover $\text{Aut}(\overline{\mathcal{X}})_e$ -equivariant.

Using the claim one can now establish the proposition. In view of Lemma 7.15 it suffices to show that \mathcal{W}_x is $\text{Aut}(\overline{\mathcal{X}})_x$ -equivariantly isomorphic to Δ as an abstract simplicial complex. Fix a flat $F \in \mathcal{F}_x$ and let $A := \varphi_x(F)$ be the corresponding twin apartment. By Corollary 7.14 there exists an isomorphism of simplicial complexes between $\mathcal{W}_x(F)$ and Δ . Fixing such an isomorphism one can extend it to an $\text{Aut}(\overline{\mathcal{X}})_x$ -equivariant family of simplicial isomorphisms $(\mathcal{W}_x(kF) \rightarrow kA)_{k \in \text{Aut}(\overline{\mathcal{X}})_x}$. To see that these identifications combine to yield the desired $\text{Aut}(\overline{\mathcal{X}})_x$ -equivariant isomorphism $\mathcal{W}_x \rightarrow \Delta$ one just needs to observe that the intersections match. However, this is clear from the fact that on both sides of the correspondence, intersections correspond to intersections of maximal tori in G . \square

In view of the proposition we refer to $\Delta_x \subset \partial_x \overline{\mathcal{X}}$ as the *twin building at the horizon of x* .

7D The global structure of the municipality

By Proposition 7.17 each fiber Δ_x of the municipality is simplicially isomorphic to the geometric realization of the twin building Δ . Combining these isomorphisms one obtains a bijection $\overline{\mathcal{X}} \times |\Delta| \rightarrow \Delta_\bullet$ which is fiberwise simplicial isomorphism. The goal of this subsection is to show that this bijection can be chosen in an $\text{Aut}(\overline{\mathcal{X}})$ -equivariant way.

Fix a basepoint $o \in \overline{\mathcal{X}}$ and an $\text{Aut}(\overline{\mathcal{X}})_o$ -equivariant isomorphism of simplicial complexes $\iota_o : \Delta \rightarrow \mathcal{W}_o$. Denote by $|\iota_o| : |\Delta| \rightarrow \Delta_o$ the induced simplicial isomorphism of realizations. We may and will always assume that ι_o is chosen in such a way that $|\Delta^+|$ is mapped to causal (rather than anti-causal) rays under ι_o . A map $\iota : \overline{\mathcal{X}} \times \Delta \rightarrow \mathcal{W}_\bullet$, (respectively, $|\iota| : \overline{\mathcal{X}} \times |\Delta| \rightarrow \Delta_\bullet$) is called an *extension* of ι_o if $\iota(o, \cdot) = \iota_o$ (respectively of $|\iota_o|$ if $|\iota|(o, \cdot) = |\iota_o|$).

Proposition 7.18. *There exists a unique $\text{Aut}(\overline{\mathcal{X}})$ -equivariant extension*

$$\iota : \overline{\mathcal{X}} \times \Delta \rightarrow \mathcal{W}_\bullet, \quad (x, C) \mapsto \iota_x(C),$$

of ι_o such that $\iota_x : \Delta \rightarrow \mathcal{W}_x$ is an isomorphism of simplicial complexes for every $x \in \overline{\mathcal{X}}$. Consequently, there exists a unique $\text{Aut}(\overline{\mathcal{X}})$ -equivariant extension

$$|\iota| : \overline{\mathcal{X}} \times |\Delta| \rightarrow \Delta_\bullet, \quad (x, p) \mapsto |\iota_x|(p),$$

of $|\iota_o|$ such that $|\iota_x| : |\Delta| \rightarrow \Delta_x$ is a simplicial isomorphism for every $x \in \overline{\mathcal{X}}$.

Proof. Given $(x, C) \in \overline{\mathcal{X}} \times \Delta$ define $\iota_x(C)$ as follows. Pick $\alpha \in \text{Aut}(\mathcal{X})$ with $\alpha(x) = o$. Via the isomorphism $\text{Aut}(\mathcal{X}) \rightarrow \text{Aut}(\text{Ad}(G))$, this α corresponds to an automorphism $\overline{\alpha}$ of $\text{Ad}(G)$. Think of a chamber $C \in \Delta$ as a Borel subgroup of $\text{Ad}(G)$; then $\overline{\alpha}(C)$ is also a Borel subgroup and hence one may define

$$\iota_x(C) := \alpha^{-1}(\iota_o(\overline{\alpha}(C))).$$

This does not depend on the choice of α . Indeed, let $\beta \in \text{Aut}(\mathcal{X})$ with $\beta(x) = o$. Then $\beta\alpha^{-1} \in \text{Aut}(\mathcal{X})_o$, and thus by $\text{Aut}(\mathcal{X})_o$ -equivariance of ι_o ,

$$\beta^{-1}(\iota_o(\overline{\beta}(C))) = \beta^{-1}(\iota_o(\overline{\beta\alpha^{-1}} \overline{\alpha}(C))) = \beta^{-1}\beta\alpha^{-1}\iota_o(\overline{\alpha}(C)) = \alpha^{-1}(\iota_o(\overline{\alpha}(C))).$$

It remains to show that ι is $\text{Aut}(\mathcal{X})$ -equivariant. For this let $(x, C) \in \overline{\mathcal{X}} \times \Delta$ and $\beta \in \text{Aut}(\overline{\mathcal{X}})$. If $\alpha \in \text{Aut}(\mathcal{X})$ satisfies $\alpha(x) = o$, then $\gamma := \alpha \circ \beta^{-1}$ satisfies $\gamma(\beta(x)) = o$. One thus gets

$$\iota_{\beta(x)}(\overline{\beta}(C)) = \gamma^{-1}(\iota_o(\overline{\gamma}(\overline{\beta}(C)))) = \beta\alpha^{-1}(\iota_o(\overline{\alpha \circ \beta^{-1}}(\overline{\beta}(C)))) = \beta(\iota_x(C)).$$

Conversely, every equivariant map extending ι_o clearly has to satisfy $\iota_x(C) := \alpha^{-1}(\iota_o(\overline{\alpha}(C)))$ for any α mapping x to o . This proves the first statement, and the second statement follows from the first. \square

Note that, if $\alpha \in \text{Aut}(\overline{\mathcal{X}})$ satisfies $\alpha(x) = y$ for some $x, y \in \overline{\mathcal{X}}$, then the diagram

$$\begin{array}{ccc} \Delta_x & \xrightarrow{\alpha} & \Delta_y \\ \uparrow |\iota_x| & & \uparrow |\iota_y| \\ |\Delta| & \xrightarrow{\alpha} & |\Delta| \end{array} \quad . \quad (7.1)$$

commutes, and this property together with the choice of $|\iota_o|$ determines $|\iota|$.

Remark 7.19. The twin building Δ is the union of its two halves Δ^\pm . Similarly, Δ_x decomposes into the subsets Δ_x^+ and Δ_x^- of causal and anti-causal rays emanating from x . Moreover, $|\iota_x|$ splits into two simplicial isomorphisms

$$|\iota_x|^+ : |\Delta^+| \rightarrow \Delta_x^+ \quad \text{and} \quad |\iota_x|^+ : |\Delta^-| \rightarrow \Delta_x^-, \quad (7.2)$$

as a consequence of the following simple observation.

Lemma 7.20. *Let $r \in \Delta_x$. Then r is causal if and only if $\iota_{r(0)}(r) \in |\Delta^+|$.*

Proof. Note that the action of the subgroup $\text{Aut}^+(\overline{\mathcal{X}})$ on Δ and Δ_\bullet preserves the two halves. Since $\text{Aut}^+(\overline{\mathcal{X}})$ acts strongly transitively on $\overline{\mathcal{X}}$ and in view of the commuting diagram (7.1) one may thus assume that $x = o$, whence the lemma follows from our choice of ι_o . \square

7E Asymptoticity of causal and anti-causal rays

Recall that two geodesic rays in a Riemannian symmetric space are called *asymptotic* if they are at bounded Hausdorff distance. For example, geodesic rays r_1, r_2 in \mathbb{E}^n are called *asymptotic* provided they are parallel and point in the same direction, i.e. they are of the form $r_1(t) = x + tv$ and $r_2(t) = y + tv$ for some $x, y \in \mathbb{R}^n$ and a unit vector v . Similarly, two geodesics rays in the hyperbolic plane \mathbb{H}^2 are asymptotic if they converge to the same point in $\partial\mathbb{H}^2 \cong S^1$. Our goal is to define similar notions of asymptoticity for causal and anti-causal rays in Kac–Moody symmetric spaces.

Again fix a basepoint $o \in \overline{\mathcal{X}}$ and an $\text{Aut}(\overline{\mathcal{X}})_o$ -equivariant isomorphism of simplicial complexes $\iota_o : \Delta \rightarrow \mathcal{W}_o$ as above. By Proposition 7.18 this gives rise to a canonical bijection

$$|\iota| : \overline{\mathcal{X}} \times |\Delta| \rightarrow \Delta_\bullet, \quad (x, p) \mapsto |\iota_x|(p),$$

which is a fiberwise simplicial isomorphism. One thus obtains for every $x, y \in \overline{\mathcal{X}}$ a unique simplicial isomorphism

$$|\iota_{x,y}| : \Delta_x \rightarrow \Delta_y$$

making the following diagram commute:

$$\begin{array}{ccc} \Delta_x & \xrightarrow{|\iota_{x,y}|} & \Delta_y \\ & \searrow |\iota_x| \quad \nearrow |\iota_y| & \\ & |\Delta| & \end{array}$$

Definition 7.21. Two rays $r_1 \in \Delta_x$ and $r_2 \in \Delta_y$ are *asymptotic*, denoted $r_1 \parallel r_2$, provided $|\iota_{x,y}|(r_1) = r_2$.

Remark 7.22. By Remark 7.19 the isomorphisms $|\iota_{x,y}| : \Delta_x \rightarrow \Delta_y$ preserve the two halves, and thus induce simplicial isomorphisms

$$|\iota_{x,y}|^\pm : \Delta_x^\pm \rightarrow \Delta_y^\pm.$$

In particular, causal rays can only be parallel to causal ray, and similarly anti-causal rays can only be parallel to anti-causal rays.

The following proposition summarizes the main properties of the equivalence relation \parallel . Concerning the statement of the proposition we observe that if $G_i < G$ is a standard rank one subgroup, then the orbit $G_i.o \subset \overline{\mathcal{X}}$ is an embedded hyperbolic plane $\mathbb{H}_{(i)}^2 \subset \overline{\mathcal{X}}$. We then refer to a subset of $\overline{\mathcal{X}}$ of the form $g.\mathbb{H}_{(i)}^2$ for some $g \in G$ and $i \in \{1, \dots, n\}$ as a *standard hyperbolic plane* in $\overline{\mathcal{X}}$.

Proposition 7.23. Let $x, y \in \overline{\mathcal{X}}$ and let $r_1 \in \Delta_x$ and $r_2 \in \Delta_y$. Then the equivalence relation \parallel satisfies the following properties:

- (A1) For every $r \in \Delta_x$ there exists a unique $r' \in \Delta_y$ with $r_1 \parallel r_2$.
- (A2) \parallel is invariant under $\text{Aut}(\overline{\mathcal{X}})$, i.e. if $r_1 \parallel r_2$, then $\alpha(r_1) \parallel \alpha(r_2)$ for all $\alpha \in \text{Aut}(\overline{\mathcal{X}})$.
- (A3) If r_1, r_2 are contained in a standard hyperbolic plane, then $r_1 \parallel r_2$ if and only if they are asymptotic in the hyperbolic sense.
- (A4) If r_1, r_2 are contained in a common maximal flat F , then $r_1 \parallel r_2$ if and only if they are asymptotic in the Euclidean sense.

Proof. (A1) is immediate from the fact that $|\iota_{x,y}|$ is a bijection.

(A2) If $r_1 \parallel r_2$, then there is a $\xi \in |\Delta|$ such that $r_1 = |\iota|(x, \xi)$ and $r_2 = |\iota|(y, \xi)$. Since $|\iota|$ is $\text{Aut}(\overline{\mathcal{X}})$ -equivariant we thus have

$$\alpha(r_1) = \alpha(|\iota|(x, \xi)) = |\iota|(\alpha(x), \alpha(\xi)) \quad \text{and} \quad \alpha(r_2) = \alpha(|\iota|(y, \xi)) = |\iota|(\alpha(y), \alpha(\xi)),$$

which implies $\alpha(r_1) \parallel \alpha(r_2)$.

(A3) By (A2) it suffices to prove the statement under the assumption that r_1 and r_2 are contained in $\mathbb{H}_{(j)}^2$ for some $j = 1, \dots, n$. One can identify $\mathbb{H}_{(j)}^2$ with the upper half-plane model \mathbb{H}^2 of the hyperbolic plane in such a way that the base point o gets identified with i . Furthermore, one can identify the image $\iota_o^{-1}(\mathbb{H}_{(j)}^2) \subset |\Delta|$ with $\mathbb{R} \cup \{\infty\}$ in such a way that $\iota_o^{-1}|_{\mathbb{H}_{(j)}^2}$ identifies geodesics in $\mathbb{H}_{(j)}^2$ emanating from o with the endpoint of the corresponding geodesic in \mathbb{H}^2 in $\mathbb{R} \cup \{\infty\}$.

Fix this identification and work in the upper half plane model from now on. If $x = i + \lambda \in \mathbb{H}^2$ for some $\lambda \in \mathbb{R}$, then an automorphism of \mathbb{H}^2 mapping o to x is given by $\tau_\lambda : z \mapsto z + \lambda$. This automorphism is induced by an element of the corresponding rank one subgroup G_j , hence extends to $\overline{\mathcal{X}}$. Given $r \in \Delta_x$ one has $\iota_x^{-1}(r) = \tau_\lambda \circ \iota_o \circ \tau_\lambda^{-1}$. In other words, $\iota_x^{-1}(r)$ is obtained by translating r by λ to the left, taking the endpoint and then shifting it by λ to the right. This,

however, is the same as just taking the endpoint of r , since this is the case for vertical geodesic rays emanating from x and the construction is equivariant with respect to the point stabilizer of x in the automorphism group. One deduces that $r \in \Delta_x \mathbb{H}$ is asymptotic to $r' \in \Delta_o$ if and only if r and r' have the same endpoint. Since every pair of points in \mathbb{H}^2 can be mapped by an automorphism of \mathbb{H}^2 to $(i, i + \lambda)$ for a suitable λ , and since any such automorphism extends to an automorphism of $\overline{\mathcal{X}}$, one deduces that our notion of asymptoticity restricts to usual hyperbolic asymptoticity on $\mathbb{H}_{(j)}^2$.

(A4) In view of (A2) one may assume that $F = \overline{A}$ is the standard maximal flat in the group model and that $x = o$. Let $\vec{\sigma}$ be the unique oriented geodesic segment from o to y and let $\tau := t[\vec{\sigma}]$ be parallel transport along $\vec{\sigma}$. Then τ acts on F as a Euclidean translation. By (7.1) one has a commuting diagram

$$\begin{array}{ccc} \Delta_o & \xrightarrow{\tau} & \Delta_y \\ \uparrow \iota_x & & \uparrow \iota_y \\ |\Delta| & \xrightarrow{\tau} & |\Delta| \end{array} \quad (7.3)$$

Now the map $\tau : |\Delta| \rightarrow |\Delta|$ is given by an element of the maximal torus $\overline{T} \subset \overline{A}$, which fixes pointwise the realization of the apartment corresponding to \overline{A} . Thus if one denotes by $\Delta_o(\overline{A})$, respectively $\Delta_y(\overline{A})$, the subsets of Δ_o and Δ_y consisting of causal or anti-causal rays in \overline{A} , then one has a commuting diagram

$$\begin{array}{ccc} \Delta_o(\overline{A}) & \xrightarrow{\tau} & \Delta_y(\overline{A}) \\ \swarrow \iota_x & & \searrow \iota_y \\ & |\Delta| & \end{array}$$

This shows that the restriction of ι_{oy} to $\Delta_o(\overline{A})$ is induced by τ , i.e. $r_o \in \Delta_o(\overline{A})$ is parallel to $r_y \in \Delta_y(\overline{A})$ if and only if r_y is obtained from r_o by a Euclidean translation, i.e. r_y is parallel to r_o in the Euclidean sense. \square

One can also describe the equivalence relation \parallel in group-theoretic terms. For this we introduce some notations concerning parabolic subgroups of \overline{G} . Given an element $\xi \in |\Delta|$, we denote by P_ξ the stabilizer of ξ in $\text{Ad}(G)$ and by $\text{supp}(\xi) \in \Delta$ the smallest simplex containing ξ . Then P_ξ is the stabilizer of $\text{supp}(\xi)$ in Δ , whence a parabolic subgroup, and in particular acts transitively on $\overline{\mathcal{X}}$ by the Iwasawa decomposition. Depending on whether $\xi \in |\Delta|^+$ or $\xi \in |\Delta|^-$ we call the parabolic P_ξ a *positive* or a *negative parabolic*. By [R  m02, Theorem 6.4.1] every parabolic subgroup P_ξ splits as a semidirect product $P_\xi = M_\xi \ltimes U_\xi$, where M_ξ is a Levi factor and U_ξ is generated by the appropriate positive root groups. Given a point $x \in \overline{\mathcal{X}}$ and $\xi \in |\Delta|$ we refer to the orbit $H_\xi(x) := U_\xi.x$ as the *horosphere* with *center* ξ through x , and we call the horosphere positive or negative according to whether $\xi \in |\Delta|^\pm$.

Proposition 7.24. *Let $x \in \overline{\mathcal{X}}$, let $r_x \in \Delta_x$ and let $\xi = \iota_x(r_x) \in |\Delta|$. Then $r \in \Delta_\bullet$ is parallel to r_x if and only if there exists $p \in P_\xi$ such that $r = pr_x$.*

Proof. Let $y \in \overline{\mathcal{X}}$, denote by K_y the stabilizer of y in G let $g \in G$ with $g.x = y$. Then $g.r_x \in \Delta_y$ and

$$\iota_y(g.r_x) = \iota_{gx}(g.r_x) = g.\iota_x(r_x) = g.\xi.$$

In particular, $\iota_y(g.r_x)$ has the same barycentric coordinates as ξ . Since K_y acts transitively on Δ and, thus, transitively on points in $|\Delta|$ with the same barycentric coordinates, there exists $k \in K_y$ such that $k.g.\xi = \xi$. Now set $p := kg$. Then $p.x = y$ and

$$\iota_y(p.r_x) = \iota_{k.y}(k.g.r_x) = k.\iota_y(r_x) = k.g.\xi = \xi.$$

Thus the unique ray $r_y \in \Delta_y$ with $r_y \parallel r_x$ is given by $r_y = p.r_x$. Since

$$p.\xi = p.\iota_x(r_x) = \iota_{p.x}(p.r_x) = \iota_y(r_y) = \xi,$$

we have $p \in P_\xi$. This shows that the asymptoticity class of r_x is contained in P_ξ . Conversely, if $r = pr_x$ for some $p \in P_\xi$ and $y := p.x$, then

$$\iota_y(r) = \iota_{p.x}(p.r_x) = p.\iota_x(r_x) = p.\xi = \xi,$$

showing that $r \parallel r_x$. \square

Remark 7.25. In conjunction with Lemma 7.20, Proposition 7.23 implies that parallelity classes of causal rays are orbits of positive parabolic subgroups, and parallelity classes of anti-causal rays are orbits of negative parabolic subgroups. In particular, parallelity classes of regular causal rays are orbits of positive Borel subgroups. Geometrically this means that one can obtain all rays parallel to a given regular causal ray r by translating r inside a flat and then sliding along a suitable positive horosphere.

7F The causal boundary

Definition 7.26. The space $\Delta_\parallel := \Delta_\bullet / \parallel$ of asymptoticity classes of causal and anti-causal rays in $\overline{\mathcal{X}}$ is called the *causal boundary* of $\overline{\mathcal{X}}_G$. Its subset $\Delta_\parallel^+ := \Delta_\bullet^+ / \parallel$ is called the *future boundary* of $\overline{\mathcal{X}}$, and the complement $\Delta_\parallel^- := \Delta_\bullet^- / \parallel$ is called the *past boundary* of $\overline{\mathcal{X}}$.

By Proposition 7.23 the $\text{Aut}(\overline{\mathcal{X}})$ -action on Δ_\bullet descends to an $\text{Aut}(\overline{\mathcal{X}})$ -action on Δ_\parallel , and similarly the subgroup $\text{Aut}^+(\overline{\mathcal{X}})$ acts on the future and the past boundary, whereas each point reflection swaps the two boundaries.

The causal boundary Δ_\parallel inherits a simplicial structure by demanding that some (hence any) of the maps

$$\Delta_x \hookrightarrow \Delta_\bullet \rightarrow \Delta_\parallel$$

is a simplicial isomorphism, and $\text{Aut}(\overline{\mathcal{X}}_G)$ acts on Δ_\parallel by simplicial automorphisms.

Corollary 7.27. (i) The causal boundary Δ_\parallel is simplicially and $\text{Aut}(\overline{\mathcal{X}}_G)$ -equivariantly isomorphic to the geometric realization $|\Delta|$ of the twin building Δ of G .

(ii) Every automorphism of $\overline{\mathcal{X}}_G$ is uniquely determined by the induced simplicial automorphism of the causal boundary.

Proof. (i) By Proposition 7.17 and Proposition 7.23 one has an $\text{Aut}(\overline{\mathcal{X}})_x$ -simplicial isomorphism

$$|\Delta| \rightarrow \Delta_x \hookrightarrow \Delta_\bullet \rightarrow \Delta_\parallel$$

for every $x \in \overline{\mathcal{X}}_G$. Since $x \in \overline{\mathcal{X}}_G$ is arbitrary and the point stabilizers $\text{Aut}(\overline{\mathcal{X}})_x$ generate $\text{Aut}(\overline{\mathcal{X}})$, it is actually $\text{Aut}(\overline{\mathcal{X}}_G)$ -equivariant. (ii) Let $\text{Aut}(\overline{\mathcal{X}}_G) \rightarrow \text{Aut}(\Delta_\parallel)$ be the homomorphism which assigns to each automorphism of $\overline{\mathcal{X}}_G$ the induced simplicial automorphism of Δ_\parallel and let $\text{Aut}(\Delta_\parallel) \rightarrow \text{Aut}(\Delta)$ be the isomorphism given by (i). Unravelling definitions one checks that the composition

$$\text{Aut}(\overline{\mathcal{X}}_G) \rightarrow \text{Aut}(\Delta_\parallel) \rightarrow \text{Aut}(\Delta)$$

coincides with the inclusion $\text{Aut}(\overline{\mathcal{X}}_G) \cong \text{Aut}(G) \rightarrow \text{Aut}(\Delta)$ given by Theorem 6.5 and Corollary 6.3. \square

We have shown Theorems 1.15 and 1.16.

8 Geometry of the causal pre-order

We keep the notation of the previous section, i.e., G denotes a simply connected centered split real Kac–Moody group satisfying the assumptions of Convention 4.2 on page 33, the group \overline{G} denotes its semisimple adjoint quotient, and $\text{Ad}(G)$ its adjoint quotient. Moreover, $\Delta = \Delta^- \sqcup \Delta^+$ denotes the common twin building associated to the canonical BN pairs of these groups. We consider the reduced Kac–Moody symmetric space $\overline{\mathcal{X}} = \overline{G}/\overline{K}$.

8A Causal curves and the causal pre-order

Definition 8.1. A *piecewise geodesic causal curve* is a causal curve $\gamma : [S, T] \rightarrow \overline{\mathcal{X}}$ with $0 < S < T < \infty$ for which there exist $S = t_0 < t_1 < \dots < t_N = T$ such that $\gamma|_{[t_i, t_{i+1}]}$ is a causal segment for every $i = 0, \dots, N-1$.

Given $x, y \in \overline{\mathcal{X}}$ write $x \prec y$ and say that x *strictly causally precedes* y if there exists a piecewise geodesic causal curve $\gamma : [S, T] \rightarrow \overline{\mathcal{X}}$ with $\gamma(S) = x$ and $\gamma(T) = y$. Write $x \preceq y$ if $x \prec y$ or $x = y$ and say that x *causally precedes* y .

By definition, \preceq is a pre-order, i.e. a reflexive and transitive relation, called the *causal pre-order* on $\overline{\mathcal{X}}$. Theorem 8.8 below implies that for a large class of Kac–Moody groups the causal pre-order is actually a partial order, i.e. anti-symmetric.

Since the group $\text{Aut}^+(\overline{\mathcal{X}})$ preserves the class of piecewise geodesic causal curves, it also preserves the causal pre-order \preceq in the sense that

$$x \preceq y \Rightarrow \alpha(x) \preceq \alpha(y) \quad (x, y \in \overline{\mathcal{X}}, \alpha \in \text{Aut}^+(\overline{\mathcal{X}})). \quad (8.1)$$

Definition 8.2. Let $x \in \overline{\mathcal{X}}$. The *strict causal future*, respectively *strict causal past* of x are defined as

$$\overline{\mathcal{X}}_x^+ := \{y \in \overline{\mathcal{X}} \mid y \succ x\} \quad \text{and} \quad \overline{\mathcal{X}}_x^- := \{y \in \overline{\mathcal{X}} \mid y \prec x\}.$$

Remark 8.3. If one denotes by $\overline{S}^\pm \subset \overline{G}$ the semigroups generated by \overline{A}_\pm and \overline{K} , then $\overline{\mathcal{X}}_e^\pm$ is simply the \overline{S}^\pm -orbit through e . Note that, by definition,

$$\overline{S}^\pm = \bigcup_{n=1}^{\infty} (\overline{K} \overline{A}_\pm \overline{K})^n$$

and that \overline{A}_+ is a subsemigroup of \overline{G} . Since \overline{S}^\pm contains \overline{K} , the semigroups \overline{S}^\pm can also be characterized as

$$\overline{S}^+ = \{g \in \overline{G} \mid g.e \succ e\} \quad \text{and} \quad \overline{S}^- = \{g \in \overline{G} \mid g.e \prec e\}$$

In particular, \preceq is a partial order if and only if $\overline{S}^+ \cap \overline{S}^- = \emptyset$.

Proposition 8.4. *Exactly one of the following two options holds in $\overline{\mathcal{X}}$:*

- (i) \preceq is a partial order and $\overline{S}^+ \cap \overline{S}^- = \emptyset$.
- (ii) $g \preceq h \preceq g$ for all $g, h \in \overline{\mathcal{X}}$ and $\overline{S}^+ = \overline{S}^- = \overline{G}$.

Proof. Obviously the two conditions are mutually exclusive. Assume that (i) fails, i.e., that \preceq is not anti-symmetric. By G -invariance one then finds $x \in \overline{\mathcal{X}}$ such that

$$e \prec x \prec e.$$

By definition, this means that there exist points $x_1, \dots, x_n = x, y_1, \dots, y_n = y$ and causal geodesic segments from e to x_1 , x_1 to x_2 , \dots , x_{n-1} to x_n and x_n to y_1 , \dots , y_{n-1} to y_n and y_n to e . In particular, $y \prec e$ is contained in a common flat F with e and the geodesic ray in F emanating

from y and through e is causal. Since K acts transitively on flats through e there exists $k \in K$ which maps F to the standard flat $\overline{A}\overline{K}$. Then $z := k*y$ has the following properties: Firstly, since $y \prec e \prec y$ and $k.e = e$ one has

$$z \prec e \prec z.$$

Moreover, z lies in \overline{A} and the geodesic ray emanating from e through z is anti-causal. In other words, $z = \exp(-X)$ for some $X \in \mathcal{C}^o$. Now consider parallel transport τ along the geodesic segment from e to z . One has $\tau(\exp(-nX)) = \exp(-(n+1)X)$ for all $n \geq 0$. Therefore $e \prec z$ implies that for all $n \geq 0$,

$$\exp(-nX) = \tau^n(e) \prec \tau^n(z) = \exp(-(n+1)X).$$

Thus transitivity of \prec yields

$$e \prec \exp(-nX) \quad \text{for all } n \geq 1,$$

and thus

$$\overline{\mathcal{X}}_e^+ \supset \bigcup_{n \geq 1} \overline{\mathcal{C}}_{\exp(-nX)}^+.$$

In particular,

$$\overline{\mathcal{X}}_e^+ \cap \overline{A}\overline{K} \supset \bigcup_{n \geq 1} (\overline{\mathcal{C}}_{\exp(-nX)}^+ \cap \overline{A}\overline{K}) \supset \bigcup_{n \geq 1} \exp(\mathcal{C}^o - nX) \overline{K} = \exp\left(\bigcup_{n \geq 1} \mathcal{C}^o - nX\right) \overline{K} = \exp(\overline{\mathfrak{a}}) \overline{K} = \overline{A}\overline{K},$$

i.e., $\overline{A}\overline{K} \subset \overline{\mathcal{X}}_e^+ = \{x \in \overline{\mathcal{X}} \mid x \succ e\}$. Since

$$\overline{S}^+ = \bigcup_{n=1}^{\infty} (\overline{K}\overline{A}_+\overline{K})^n = \{g \in \overline{G} \mid g.e \succ e\},$$

the semigroup \overline{S}^+ contains \overline{A} and \overline{K} . It therefore contains each of the finite products $\overline{K}\overline{A}\overline{K} \cdots \overline{A}\overline{K}$. Proposition 5.13 implies $\overline{\mathcal{X}}_e^+ = \overline{\mathcal{X}}$, i.e. $x \succ e$ for all $x \in \overline{\mathcal{X}}$, and thus (ii) holds. \square

8B Convexity conditions

In this section we formulate a condition on the group Kac–Moody group G which will ensure that the pre-order \preceq on $\overline{\mathcal{X}}$ is anti-symmetric, i.e., a partial order. Given a root group $U_\beta \subset G$ denote by \overline{U}_β its image in \overline{G} . Note that the map $U_\beta \rightarrow \overline{U}_\beta$ is an isomorphism.

Definition 8.5. The Kac–Moody group G is called *rank one convex* if for any $a \in \overline{A}_+$, any simple root $\alpha_i \in \Pi$ and any $k \in \overline{K}_{\alpha_i} := \overline{G}_{\alpha_i} \cap \overline{K}$ there exists $u \in \overline{U}_{\alpha_i}$ such that

$$uka\overline{K} \in \overline{A}_+\overline{K}.$$

See Theorem 8.9 below for a relation between this condition and the classical Kostant convexity theorem [Kos73, Theorem 4.1]. The following result states that rank one convexity implies that the causal pre-order is anti-symmetric.

Theorem 8.6. *Let the (non-spherical, non-affine) Kac–Moody group G be rank one convex. Then the following hold:*

(i) *The semigroup \overline{S}^+ satisfies*

$$\overline{S}^+ \subset \overline{U}_+\overline{A}_+\overline{K}.$$

(ii) *In the group model,*

$$\overline{\mathcal{X}}_e^+ \subset \tau(\overline{U}_+\overline{A}_+).$$

(iii) The causal pre-order \preceq is a partial order.

Proof. Clearly (i) and (ii) are equivalent, and if (i) holds, then $\overline{S}^+ \subsetneq \overline{G}$ by the Iwasawa decomposition (Theorem 3.23), and thus (iii) holds by Proposition 8.4. It thus suffices to show (i).

Let $g \in \overline{S}^+$. Since \overline{K} is generated by $\overline{K}_{\alpha_1}, \dots, \overline{K}_{\alpha_n}$ there exist $\beta_1, \dots, \beta_t \in \Pi$ and elements $k_i \in \overline{K}_{\beta_i}$ and $a_i \in \overline{A}_+$ such that

$$\tau(g) = \tau(k_1 a_1 \cdots k_t a_t).$$

By rank one convexity of G there exists $u_i \in U_{\beta_i}$ such that $a'_i \overline{K} := u_i^{-1} k_i a_i \overline{K} \in \overline{A}_+ \overline{K}$ and thus

$$\tau(g) = \tau(u_1 a'_1 \cdots u_t a'_t).$$

Since \overline{A} normalizes every real root group there are $u'_i \in U_{\beta_i}$ such that

$$\tau(g) = \tau(u'_1 \cdots u'_t a'_1 \cdots a'_t).$$

Since \overline{U}_+ and \overline{A}_+ are semigroups one thus has $\tau(g) \in \overline{U}_+ \overline{A}_+$ and thus $g \in \overline{U}_+ \overline{A}_+ \overline{K}$. This finishes the proof. \square

8C Causal partial orders on star-spherical Kac–Moody symmetric spaces

By Theorem 8.6, rank one convexity of a non-spherical non-affine irreducible Kac–Moody group implies the existence of a non-trivial partial order on its Kac–Moody symmetric space. The following class of Kac–Moody groups has this property by a straightforward local-to-global argument using classical Kostant convexity of Lie groups as stated in [Kos73, Theorem 4.1].

Definition 8.7. Let G be a simply connected centered split real Kac–Moody group satisfying the assumptions of Convention 4.2 on page 33, let \mathbf{A} be the underlying Cartan matrix and let $\Gamma_{\mathbf{A}}$ be the associated Dynkin diagram. Then G and $\overline{\mathcal{X}}$ are called *star-spherical* if for any vertex v of $\Gamma_{\mathbf{A}}$ the star $\text{st}(v)$ of v in $\Gamma_{\mathbf{A}}$ is a Dynkin diagram of spherical type.

Note that this notion actually depends only on the underlying Coxeter system (W, S) since it is equivalent to the property that the vertex stars in the Coxeter diagram are of spherical type. Notable examples of irreducible non-spherical non-affine star-spherical Kac–Moody groups include the E_n -series ($n \geq 10$) and the AE_n series ($n \geq 5$). Moreover, among the non-compact hyperbolic Coxeter groups, all but three are star-spherical. The purpose of this section is to prove the following result:

Theorem 8.8. *Every star-spherical Kac–Moody group G is rank one convex. In particular, the causal pre-order on the reduced symmetric space of a star-spherical Kac–Moody group is a partial order.*

Let H a split real quasi-simple Lie group H . Consider H as a split real Kac–Moody group of spherical type and use the notation for subgroups of H as introduced for subgroups of G . In particular, denote by $H = U_H^+ A_H K_H$ an Iwasawa decomposition of H . Denote by \mathfrak{h} , respectively \mathfrak{a}_H the Lie algebras of H and A_H and by $\Phi_H \subset \mathfrak{a}_H^*$ the set of restricted roots of \mathfrak{h} with respect to \mathfrak{a}_H . Also denote by $W_H = N_H(A_H)/Z_H(A_H)$ the Weyl group of H . Given a subset $X \subset A_H$ denote by $\text{conv}(X)$ its geodesic convex hull, i.e.

$$\text{conv}(X) := \exp(\text{conv}(\log(X))).$$

The following is a restatement of Kostant’s convexity theorem.

Theorem 8.9 (Kostant). *Let $a \in A_H$ and $k \in K_H$. Then there exist a unique $a' \in \text{conv}(W_H \cdot a)$ and a unique $u \in U_H^+$ such that*

$$ukaK_H = a'K_H.$$

Moreover, if α is a simple root of H and $k \in K_{\alpha}$, then $u \in U_{\alpha}$ and $a' \in \text{conv}(W_{\langle U_{\alpha}, U_{-\alpha} \rangle} \cdot a)$.

Proof. By the Iwasawa decomposition $H = U_H^+ \times A_H \times K_H$ (Theorem 3.23) there exist unique $u^{-1} \in U_H^+$, $a' \in A_H$ and $k' \in K_H$ such that $ka = u^{-1}a'k'$ and, thus, $ukaK = a'K$. Kostant's classical convexity theorem [Kos73, Theorem 4.1] states that $a' \in \text{conv}(W_H.a)$. If $k \in K_\alpha$, then necessarily $u \in U_\alpha$ because of the uniqueness of the Iwasawa decomposition (Theorem 3.23). Indeed, let $H_\alpha := \langle U_\alpha, U_{-\alpha} \rangle$ and decompose $a = a_\alpha a^\alpha$ with $a_\alpha \in H_\alpha \cap A_H$ and $a^\alpha \in A_H$ centralized by G_α ; the latter choice is possible since the group $H_\alpha A_H$ is a reductive Lie group. By the first assertion of the theorem applied to H_α there exists a unique $u_\alpha \in U_\alpha$ such that $u_\alpha k a_\alpha K_\alpha = a'_\alpha K_\alpha$ and hence

$$u_\alpha k a K_\alpha = a^\alpha u_\alpha k a_\alpha K_\alpha = a^\alpha a'_\alpha K_\alpha.$$

Because of the above-mentioned uniqueness of the Iwasawa decomposition in H one has $u = u_\alpha \in U_\alpha$ and

$$a' = a^\alpha a'_\alpha \in a^\alpha \text{conv}(W_{H_\alpha}.a_\alpha) = \text{conv}(W_{H_\alpha}.a). \quad \square$$

Proof of Theorem 8.8. We work in the coset model. Then Kostant convexity amounts to showing that for every positive simple root α ,

$$\forall a \in \overline{A}_+ \forall k \in \overline{K}_\alpha \exists u \in \overline{U}_\alpha : uka\overline{K} \in \overline{A}_+\overline{K}. \quad (8.2)$$

Denote by S_α the vertices in the star of α in $\Gamma_{\mathbf{A}}$. If $\beta \in \Pi \setminus S_\alpha$, then \overline{G}_α centralizes \overline{G}_β . Denote by $H < \overline{G}$ the subgroup generated by the family of rank one groups $(\overline{G}_\beta)_{\beta \in S_\alpha}$. By hypothesis, H is a quasi-simple Lie group to which Theorem 8.9 applies. Write $a = a_H a'$ where $a_H \in A_H$ and $a' \in \langle (\overline{G}_\beta)_{\beta \in \Pi \setminus S_\alpha} \rangle$; note that a' is centralized by \overline{G}_α and fixed by the Weyl group $W_{\overline{G}_\alpha}$. By Theorem 8.9 there exists $u \in \overline{U}_\alpha$ such that $uka_H K_H \in \text{conv}(W_{\overline{G}_\alpha}.a_H)K_H$ and thus

$$uka\overline{K} = a'uka_H K_H \overline{K} \in a' \text{conv}(W_{\overline{G}_\alpha}.a_H)K_H \overline{K} = \text{conv}(W_{\overline{G}_\alpha}.a)\overline{K}.$$

Since $a \in \overline{A}_+$ and the Tits cone is convex and invariant under the Weyl group, this implies (8.2) and finishes the proof. \square

We have established Theorem 1.17.

A Complex Kac–Moody algebras and the Weyl group

AA Complex Kac–Moody algebras

Let \mathbf{A} be a generalized Cartan matrix in the sense of Definition 3.2 (see also [Kac90, §1.1]). Then one can associate to \mathbf{A} several complex Lie algebras as follows.

In [Kac90, §1.3] Kac defines a quadruple

$$(\mathfrak{g}(\mathbf{A}), \mathfrak{h}(\mathbf{A}), \Pi, \check{\Pi}) \quad (\text{A.1})$$

consisting of a complex Lie algebra $\mathfrak{g}(\mathbf{A})$, an abelian subalgebra $\mathfrak{h}(\mathbf{A})$ and finite subsets $\Pi = \{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{h}(\mathbf{A})^*$ and $\check{\Pi} = \{\check{\alpha}_1, \dots, \check{\alpha}_n\} \subset \mathfrak{h}(\mathbf{A})$ called *simple roots* and *simple coroots* respectively. A useful characterization of this quadruple $(\mathfrak{g}(\mathbf{A}), \mathfrak{h}(\mathbf{A}), \Pi, \check{\Pi})$ is given in [Kac90, Proposition 1.4]. In the present article $\mathfrak{g}(\mathbf{A})$ is called the *complex Kac–Moody algebra* associated with \mathbf{A} . If one denotes by

$$\Delta := \left\{ \alpha \in \sum_{i=1}^n \mathbb{Z}\alpha_i \mid \mathfrak{g}_\alpha \neq \{0\} \right\}$$

the set of $\mathfrak{h}(\mathbf{A})$ -roots in $\mathfrak{g}(\mathbf{A})$, then by [Kac90, (1.3.1)] one has the *root space decomposition*

$$\mathfrak{g}(\mathbf{A}) = \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha. \quad (\text{A.2})$$

Denote by $\mathfrak{g}_i < \mathfrak{g}(\mathbf{A})$ the complex subalgebra generated by the root spaces \mathfrak{g}_{α_i} and $\mathfrak{g}_{-\alpha_i}$. By [Kac90, (1.3.3), (1.4.1), (1.4.2)] one has

$$\mathfrak{g}_i = \langle \mathfrak{g}_{\alpha_i}, \mathfrak{g}_{-\alpha_i} \rangle \cong \mathfrak{sl}(2, \mathbb{C}).$$

Given $I \subset \{1, \dots, n\}$, define

$$\mathfrak{g}_I := \langle \mathfrak{g}_i \mid i \in I \rangle$$

and call \mathfrak{g}_I a *standard rank $|I|$ subalgebra* of $\mathfrak{g}(\mathbf{A})$.

The main object of interest in this appendix is the derived subalgebra

$$\mathfrak{g} := [\mathfrak{g}(\mathbf{A}), \mathfrak{g}(\mathbf{A})] < \mathfrak{g}(\mathbf{A}), \quad (\text{A.3})$$

which is called the *derived complex Kac–Moody algebra* associated with \mathbf{A} . It is denoted by $\mathfrak{g}'(\mathbf{A})$ in [Kac90, §1.3]. The Lie algebra \mathfrak{g} contains all the standard rank one subalgebras, as $\mathfrak{sl}(2, \mathbb{C})$ is perfect, and in fact is generated by these by [Kac90, Proposition 1.4]. The intersection

$$\mathfrak{h} := \mathfrak{h}(\mathbf{A}) \cap \mathfrak{g} = \sum_{i=1}^n \mathbb{C}\check{\alpha}_i \quad (\text{A.4})$$

is given by the complex span of the simple coroots, see [Kac90, §1.3] (where it is denoted by \mathfrak{h}'). By [Kac90, Proposition 1.6] the Lie algebra \mathfrak{h} contains the center of $\mathfrak{g}(\mathbf{A})$ and of \mathfrak{g} , which is given by

$$\mathfrak{z}(\mathfrak{g}(\mathbf{A})) = \mathfrak{z}(\mathfrak{g}) = \mathfrak{c} := \{h \in \mathfrak{h}(\mathbf{A}) \mid \forall i = 1, \dots, n : \alpha_i(h) = 0\}. \quad (\text{A.5})$$

The third Lie algebra of interest in this appendix is the quotient

$$\bar{\mathfrak{g}} := \mathfrak{g}/\mathfrak{c},$$

called the *adjoint complex Kac–Moody algebra* associated with \mathbf{A} . Since $\mathfrak{sl}(2, \mathbb{C})$ is simple, the standard rank one subalgebras \mathfrak{g}_i embed into $\bar{\mathfrak{g}}$ and so do in fact all root spaces \mathfrak{g}_α , $\alpha \neq 0$, from (A.2), whereas the image of $\mathfrak{g}_0 = \mathfrak{h}$ in $\text{ad}(\mathfrak{g})$ is given by $\bar{\mathfrak{h}} := \mathfrak{h}/\mathfrak{c}$. By definition one has

$$\mathfrak{h}(\mathbf{A}) \hookrightarrow \mathfrak{h} \twoheadrightarrow \bar{\mathfrak{h}}.$$

If \mathbf{A} is of size $n \times n$ and of rank l , then the complex dimensions of the abelian subalgebras are given by

$$\dim_{\mathbb{C}} \mathfrak{h}(\mathbf{A}) = 2n - l, \quad \dim_{\mathbb{C}} \mathfrak{h} = n, \quad \dim_{\mathbb{C}} \bar{\mathfrak{h}} = l, \quad (\text{A.6})$$

cf. [Kac90, (1.1.3), resp. §1.3, resp. Proposition 1.6]. In particular, on one hand one has the following:

Observation A.1. *Let \mathbf{A} be an invertible generalized Cartan matrix. Then*

$$\mathfrak{g}(\mathbf{A}) = \mathfrak{g} = \bar{\mathfrak{g}}.$$

The following example on the other hand illustrates the differences between the Lie algebras $\mathfrak{g}(\mathbf{A})$, \mathfrak{g} and $\bar{\mathfrak{g}}$ for an irreducible generalized Cartan matrix of affine type.

Example A.2. Let \mathbf{A} be an irreducible generalized Cartan matrix of affine type and denote by $\overset{\circ}{\mathfrak{g}}$ the finite-dimensional simple Lie algebra associated with the corresponding Cartan matrix of finite type. Then in the notation of [Kac90, Chapter 7] the Lie algebra

$$\bar{\mathfrak{g}} = \mathcal{L}(\overset{\circ}{\mathfrak{g}})$$

is the loop algebra of $\overset{\circ}{\mathfrak{g}}$, whereas

$$\mathfrak{g} = \mathcal{L}(\overset{\circ}{\mathfrak{g}}) \oplus \mathbb{C}K$$

is a one-dimensional central extension of the loop algebra and

$$\mathfrak{g}(\mathbf{A}) = \mathcal{L}(\overset{\circ}{\mathfrak{g}}) \oplus \mathbb{C}K \oplus \mathbb{C}d$$

for a certain derivation d . The complex dimensions of $\mathfrak{h}(\mathbf{A})$, \mathfrak{h} and $\bar{\mathfrak{h}}$ are given by $\text{rk}(\overset{\circ}{\mathfrak{g}}) + 2$, $\text{rk}(\overset{\circ}{\mathfrak{g}}) + 1$ and $\text{rk}(\overset{\circ}{\mathfrak{g}})$, respectively.

For a symmetrizable generalized Cartan matrix \mathbf{A} as defined in [Kac90, §2.1] (see also Definition 3.2) the Gabber–Kac Theorem provides a very efficient way of defining the Kac–Moody algebra $\mathfrak{g}(\mathbf{A})$ and the derived Kac–Moody algebra \mathfrak{g} . The statement for the derived Kac–Moody algebra is as follows.

Theorem A.3 (Gabber–Kac Theorem). *Let $\mathbf{A} = (a_{ij})_{1 \leq i, j \leq n}$ be a symmetrizable generalized Cartan matrix of size $n \times n$. Then the derived complex Kac–Moody algebra \mathfrak{g} is isomorphic to the quotient of the free complex Lie algebra on $3n$ generators $e_i, f_i, h_i, 1 \leq i \leq n$, modulo the following relations:*

$$\begin{aligned} [h_i, h_j] &= 0, & [e_i, f_i] &= h_i, & [e_i, f_j] &= 0 \quad (i \neq j), \\ [h_i, e_j] &= a_{ij}e_j, & [h_i, f_j] &= -a_{ij}f_j, \\ (\text{ade}_i)^{1-a_{ij}}e_j &= 0 \quad (i \neq j), & (\text{ade}_i)^{1-a_{ij}}f_j &= 0 \quad (i \neq j), \end{aligned}$$

via the homomorphism that maps h_i to $\check{\alpha}_i$ and transforms a_{ij} into $\alpha_j(\check{\alpha}_i)$.

In particular, \mathfrak{g} is the colimit of the amalgam of Lie algebras consisting of its standard subalgebras $\mathfrak{g}_i, \mathfrak{g}_{i,j}$ of rank one and two.

Proof. See [GK81], [Kac90, Theorem 9.11] plus [Kac90, Remark 1.5]. \square

The presentation of \mathfrak{g} from the preceding theorem is called the *Gabber–Kac presentation*. Of course, one obtains a presentation of $\bar{\mathfrak{g}}$ by adding the elements of \mathfrak{c} as relators to the Gabber–Kac presentation.

Notation A.4. Since $\mathfrak{h} = \sum_{i=1}^n \mathbb{C} \check{\alpha}_i$, one can define a real form \mathfrak{a} of \mathfrak{h} by setting

$$\mathfrak{a} := \text{span}_{\mathbb{R}}(\check{\alpha}_1, \dots, \check{\alpha}_n).$$

Dually, also define a subspace $V \subset \mathfrak{h}(\mathbf{A})^*$ by

$$V := \text{span}_{\mathbb{R}}(\alpha_1, \dots, \alpha_n)$$

Then the image of \mathfrak{a} under the canonical projection $\mathfrak{h} \rightarrow \bar{\mathfrak{h}}$ defines a real form of $\bar{\mathfrak{h}}$ which is denoted by $\bar{\mathfrak{a}}$. One then has the following commutative diagram, where all maps are the obvious inclusions/projections, respectively their dual maps:

$$\begin{array}{ccc} & \mathfrak{h}(\mathbf{A}) & \\ & \uparrow \iota_{\mathbb{C}} & \\ \mathfrak{a} & \xrightarrow{j} & \mathfrak{h} \\ \pi \downarrow & & \downarrow \pi_{\mathbb{C}} \\ \bar{\mathfrak{a}} & \xrightarrow{\quad} & \bar{\mathfrak{h}} \end{array} \qquad \begin{array}{ccc} & \mathfrak{h}(\mathbf{A})^* & \xleftarrow{\quad} V \\ & \downarrow \iota_{\mathbb{C}}^* & \downarrow \\ \mathfrak{a}^* & \xleftarrow{j^*} & \mathfrak{h}^* \xleftarrow{\quad} \iota_{\mathbb{C}}^*(V) \\ \pi^* \uparrow & & \uparrow \pi_{\mathbb{C}}^* \\ \bar{\mathfrak{a}}^* & \xleftarrow{\quad} & \bar{\mathfrak{h}}^* \end{array} \tag{A.7}$$

All of these maps are linear (over \mathbb{R} and \mathbb{C} respectively) and injective/surjective as indicated by the arrows. By Proposition A.6 in fact $\pi^*(\bar{\mathfrak{a}}^*) \cong \iota_{\mathbb{C}}^*(V)$.

AB The Kac–Moody representation of the Weyl group

Recall that a [Coxeter system](#) is a pair (W, S) consisting of a group W and a (finite) generating system $S = \{r_1, \dots, r_n\}$ such that

$$W = \langle r_1, \dots, r_n \mid r_i^2 = 1, (r_i r_j)^{m_{ij}} = 1 \rangle$$

for suitable $(m_{ij})_{i,j} \subset \mathbb{Z} \cup \{\infty\}$ is a presentation of W by generators and relations. The matrix $M = (m_{ij})_{i,j}$ is called the [Coxeter matrix](#) of the Coxeter system (W, S) .

Definition A.5. Following [Kac90, §3.7], given $i \in \{1, \dots, n\}$ define $r_{\alpha_i} \in \text{GL}(\mathfrak{h}(\mathbf{A})^*)$ by

$$r_{\alpha_i}(\lambda) = \lambda - \lambda(\check{\alpha}_i)\alpha_i; \tag{A.8}$$

dually, define $\check{r}_{\alpha_i} \in \text{GL}(\mathfrak{h}(\mathbf{A}))$ by

$$\check{r}_{\alpha_i}(h) = h - \alpha_i(h)\check{\alpha}_i. \tag{A.9}$$

The groups $W := \langle \check{r}_{\alpha_1}, \dots, \check{r}_{\alpha_n} \rangle < \text{GL}(\mathfrak{h}(\mathbf{A}))$ and $\langle r_{\alpha_1}, \dots, r_{\alpha_n} \rangle < \text{GL}(\mathfrak{h}(\mathbf{A})^*)$ are canonically isomorphic via $\check{r}_{\alpha_i} \mapsto r_{\alpha_i}$; the group W is called the [Weyl group](#) associated with the generalized Cartan matrix \mathbf{A} . The Coxeter diagram induced in Definition 3.2 from the Dynkin diagram of \mathbf{A} in fact exactly contains the relevant information on the m_{ij} , $i \neq j$, from the Coxeter matrix.

With the notation introduced in Notation A.4 one has:

Proposition A.6. (i) The action of the Weyl group W defined in (A.9) stabilizes the complex subalgebra $\mathfrak{h} < \mathfrak{h}(\mathbf{A})$ and its real form \mathfrak{a} , acts trivially on \mathfrak{c} and thus induces actions of W on $\bar{\mathfrak{h}}$ and $\bar{\mathfrak{a}}$.

(ii) The action of the Weyl group W defined in (A.8) stabilizes the real subspace $V < \mathfrak{h}(\mathbf{A})^*$.

(iii) The map j^* induces an isomorphism

$$\iota_{\mathbb{C}}^*(V) \xrightarrow{\cong} \pi^*(\bar{\mathfrak{a}}^*)$$

(iv) The action of the Weyl group W from assertion (ii) acts trivially on $\ker(\iota^*)$ and, thus, induces an action of W on $\pi^*(\bar{\mathfrak{a}}^*) \cong \iota_{\mathbb{C}}^*(V)$ and, by transport of structure, on $\bar{\mathfrak{a}}^*$.

Proof. It is immediate from (A.8) and (A.9) that each r_{α_i} maps simple roots to \mathbb{R} -linear combinations of simple roots and each \tilde{r}_{α_i} maps simple coroots to \mathbb{R} -linear combinations of simple coroots. Moreover, each r_{α_i} acts trivially on \mathfrak{c} by (A.5) and (A.9). Assertions (i) and (ii) follow.

In order to prove (iii) recall from (A.6) that the quotient $\bar{\mathfrak{a}}$ has \mathbb{R} -dimension l , and so do $\bar{\mathfrak{a}}^*$ and $\pi^*(\bar{\mathfrak{a}}^*)$. For each $h \in \mathfrak{c} \cap \mathfrak{a}$ one has $(\iota^*(\alpha_i))(h) = (\alpha_i \circ \iota)(h) = (\alpha_i)_{\mathfrak{a}}(h) = 0$. That is, each $\iota^*(\alpha_i) = \alpha_i|_{\mathfrak{a}}$ in fact is of the form $\pi^*(\bar{\alpha}_i) = \bar{\alpha}_i \circ \pi$ for a uniquely determined $\bar{\alpha}_i \in \bar{\mathfrak{a}}^*$; in other words, $\iota^*(\alpha_i) \in \pi^*(\bar{\mathfrak{a}}^*)$. Since V equals the \mathbb{R} -span of the simple roots $\alpha_1, \dots, \alpha_n$, the image $\iota^*(V)$ equals the \mathbb{R} -span of the images $\iota^*(\alpha_1), \dots, \iota^*(\alpha_n)$. In particular, $\iota^*(V) \leq \pi^*(\bar{\mathfrak{a}}^*)$.

Since \mathfrak{a} is the \mathbb{R} -span of the simple coroots $\check{\alpha}_1, \dots, \check{\alpha}_n$, the \mathbb{R} -dimension of the image $\iota^*(V)$ equals the rank of the generalized Cartan matrix \mathbf{A} , i.e., $\dim_{\mathbb{R}}(\iota^*(V)) = l$. One concludes $\pi^*(\bar{\mathfrak{a}}^*) = \iota^*(V)$.

In order to prove (iv), observe that $\lambda \in \ker(\iota^*)$ if and only if for each $1 \leq i \leq n$ one has $\lambda(\check{\alpha}_i) = 0$. Therefore for any $\lambda \in \ker(\iota^*)$ one has $r_{\alpha_i}(\lambda) = \lambda - \lambda(\check{\alpha}_i)\alpha_i = \lambda$ by (A.8); that is, W acts trivially on $\ker(\iota^*)$ \square

Definition A.7. The real representations

$$\rho_{KM} : W \rightarrow \mathrm{GL}(\mathfrak{a}) \quad \text{and} \quad \bar{\rho}_{KM} : W \rightarrow \mathrm{GL}(\bar{\mathfrak{a}})$$

defined by (i) are called the *Kac–Moody representation* of W and the *reduced Kac–Moody representation* of W , respectively. The real representation

$$\bar{\rho}_{KM} : W \rightarrow \mathrm{GL}(\bar{\mathfrak{a}}^*)$$

defined by (iv) is called the *dual reduced Kac–Moody representation*.

AC The Coxeter system of the Weyl group

For $S := \{r_{\alpha_1}, \dots, r_{\alpha_n}\}$ the pair (W, S) is a Coxeter system by [Kac90, §3.13]. According to [Kac90, Proposition 3.13] (see also Definition 3.2) its Coxeter matrix $M = (m_{ij})_{i,j}$ is given by $m_{ii} = 1$ and m_{ij} for $i \neq j$ by

$$m_{ij} = \begin{cases} 2, & a_{ij}a_{ji} = 0, \\ 3, & a_{ij}a_{ji} = 1, \\ 4, & a_{ij}a_{ji} = 2, \\ 6, & a_{ij}a_{ji} = 3, \\ \infty, & a_{ij}a_{ji} \geq 4; \end{cases}$$

recall here from [Kac90, (1.1.2)] that $a_{ij} = \alpha_j(\check{\alpha}_i)$.

The action of the Weyl group W on $\mathfrak{h}(\mathbf{A})^*$ defined in (A.8) preserves the set Δ of $\mathfrak{h}(\mathbf{A})$ -roots in $\mathfrak{g}(\mathbf{A})$, and the elements of $\Phi = W \cdot \Pi \subset \Delta$ are called the *real roots* of $\mathfrak{g}(\mathbf{A})$. The tuple (W, S, Φ, Π) is called the *Coxeter datum* associated with the generalized Cartan matrix \mathbf{A} . Note that the Coxeter datum determines uniquely a system $\Phi^+ \subset \Phi$ of *positive roots* by demanding that Φ^+ contains Π .

To a real root $\alpha = w \cdot \alpha_i \in \Phi$, $w \in W$, corresponds the *root reflection* $\tilde{r}_{\alpha} := w \tilde{r}_{\alpha_i} w^{-1} \in W$, which depends only on α ; see [Kac90, proof of Lemma 3.10]. Note that the kernel of $\rho_{KM}(\tilde{r}_{\alpha})$ equals the hyperplane $H_{\alpha} := \ker(\alpha|_{\mathfrak{a}})$. Moreover, by (A.5), one has

$$\mathfrak{c} \cap \mathfrak{a} = \bigcap_{\alpha \in \Phi} H_{\alpha} = \bigcap_{i=1}^n H_{\alpha_i}.$$

In particular, the image $\bar{H}_{\alpha} := \pi(H_{\alpha}) \subset \bar{\mathfrak{a}}$ is a hyperplane in $\bar{\mathfrak{a}}$.

Definition A.8. The hyperplane $H_\alpha \subset \mathfrak{a}$ (and by abuse of language also its image $\overline{H}_\alpha \subset \overline{\mathfrak{a}}$) is called the *root hyperplane* associated with α . The union of the root hyperplanes in $\overline{\mathfrak{a}}$ is denoted

$$\overline{\mathfrak{a}}^{\text{sing}} := \bigcup_{\alpha \in \Phi} \overline{H}_\alpha,$$

its complement by

$$\overline{\mathfrak{a}}^{\text{reg}} := \overline{\mathfrak{a}} \setminus \overline{\mathfrak{a}}^{\text{sing}}.$$

Elements of $\overline{\mathfrak{a}}^{\text{sing}}$ and $\overline{\mathfrak{a}}^{\text{reg}}$ are called *singular* and *regular points* respectively. Connected components of $\overline{\mathfrak{a}}^{\text{reg}}$ are called the *geometric chambers* of $\overline{\mathfrak{a}}$.

AD Root bases and Coxeter systems

Recall from [Kra09, Definition 1.2.1] that a triple $(E, (-|-, \Pi)$ is called a *root basis* if E is real vector space, $(-|-,)$ is a symmetric bilinear form on E and $\Pi \subset E$ is a finite set such that the following hold:

- (i) For every $\xi \in \Pi$ one has $(\xi|\xi) = 1$.
- (ii) For any pair of distinct $\xi_1, \xi_2 \in \Pi$ one has

$$(\xi_1|\xi_2) \in \{-\cos(\pi/m) \mid m \in \mathbb{N}\} \cup (-\infty, -1].$$

- (iii) There exists $\lambda \in E^*$ such that $\lambda(\xi) > 0$ for all $\xi \in \Pi$.

Note that Π need not be linearly independent and $(-|-,)$ may be degenerate.

Proposition A.9 ([Kra09, Theorem 1.2.2]). *Let $(E, (-|-, \Pi)$ be a root basis, for each $\xi \in \Pi$ define $s_\xi \in \text{GL}(E)$ via*

$$s_\xi(v) = v - 2(\xi|v)\xi,$$

let $\overline{S} := \{s_\xi \mid \xi \in \Pi\}$, and let $\overline{W} := \langle \overline{S} \rangle < \text{GL}(E)$. Then $(\overline{W}, \overline{S})$ is a Coxeter system and $\overline{W} < \text{O}(E, (-|-,))$ is a discrete subgroup. \square

In the situation of the proposition $(E, (-|-, \Pi)$ is called a root basis for \overline{W} in E .

AE Root bases for Weyl groups with symmetrizable generalized Cartan matrix

In general, given a generalized Cartan matrix \mathbf{A} , one cannot find a root basis for $\rho_{KM}(W)$ in \mathfrak{a} or for $\overline{\rho}_{KM}(W)$ in $\overline{\mathfrak{a}}$. For instance, if \mathbf{A} is not symmetrizable, then there simply does not exist a suitable W -invariant bilinear form on \mathfrak{a} . Non-symmetrizability actually is the only obstruction for the existence of a root basis in \mathfrak{a} . The case of the quotient $\overline{\mathfrak{a}}$ is a bit more subtle; however, if one excludes the affine case, it is also possible to construct a root basis for W in $\overline{\mathfrak{a}}$.

For a symmetrizable generalized Cartan matrix \mathbf{A} and a diagonal matrix $D = \text{diag}(\varepsilon_1, \dots, \varepsilon_n)$ with positive entries such that $D^{-1}\mathbf{A} = (b_{ij})$ is symmetric, following [Kac90, (2.1.4)] one defines an *invariant symmetric bilinear form* on \mathfrak{a} via

$$(\check{\alpha}_i|\check{\alpha}_j) := b_{ij}\varepsilon_i\varepsilon_j.$$

Note that $b_{jj}\varepsilon_j = a_{jj} = 2$, whence

$$\begin{aligned} \check{r}_{\alpha_j}(\check{\alpha}_i) &\stackrel{(\text{A.9})}{=} \check{\alpha}_i - \alpha_j(\check{\alpha}_i)\check{\alpha}_j = \check{\alpha}_i - a_{ij}\check{\alpha}_j \\ &= \check{\alpha}_i - \varepsilon_i b_{ij}\check{\alpha}_j = \check{\alpha}_i - 2 \frac{b_{ij}\varepsilon_i\varepsilon_j}{b_{jj}\varepsilon_j^2} \check{\alpha}_j = \check{\alpha}_i - 2 \frac{(\check{\alpha}_j|\check{\alpha}_i)}{(\check{\alpha}_j|\check{\alpha}_j)} \check{\alpha}_j, \end{aligned} \tag{A.10}$$

i.e., $\check{r}_{\alpha_j}|_{\mathfrak{a}}$ is the $(-|-)$ -orthogonal reflection associated with $\check{\alpha}_j$, in particular $(-|-)$ is invariant under the action of W on \mathfrak{a} .

Define the *normalized coroots* by

$$\check{n}_j := \frac{\check{\alpha}_j}{(\check{\alpha}_j|\check{\alpha}_j)^{\frac{1}{2}}} = \frac{1}{\sqrt{2\varepsilon_j}}\check{\alpha}_j$$

and set $\check{\Pi}_{\text{nor}} := \{\check{n}_1, \dots, \check{n}_n\}$.

Following [Kac90, (2.1.6)], one dually defines an *invariant symmetric bilinear form* on V via

$$(\alpha_i|\alpha_j) := b_{ij} = \frac{a_{ij}}{\varepsilon_i}.$$

As above one computes

$$\begin{aligned} r_{\alpha_j}(\alpha_i) &\stackrel{\text{(A.8)}}{=} \alpha_i - \alpha_i(\check{\alpha}_j)\alpha_j = \alpha_i - a_{ji}\alpha_j \\ &= \alpha_i - b_{ji}\varepsilon_j\alpha_j = \alpha_i - 2\frac{b_{ji}\varepsilon_j}{a_{jj}}\alpha_j = \alpha_i - 2\frac{(\alpha_j|\alpha_i)}{(\alpha_j|\alpha_j)}\alpha_j. \end{aligned} \quad (\text{A.11})$$

Define the *normalized roots* by

$$n_j := \frac{\alpha_j}{(\alpha_j|\alpha_j)^{\frac{1}{2}}} = \frac{\sqrt{\varepsilon_j}}{\sqrt{2}}\alpha_j$$

and set $\Pi_{\text{nor}} := \{n_1, \dots, n_n\}$.

Proposition A.10. *Let \mathbf{A} be a symmetrizable irreducible generalized Cartan matrix. Then the triples*

$$(\mathfrak{a}, (-|-), \check{\Pi}_{\text{nor}}) \quad \text{and} \quad (V, (-|-), \Pi_{\text{nor}})$$

are root bases. If \mathbf{A} is non-affine, then also their images

$$(\pi(\mathfrak{a}) = \bar{\mathfrak{a}}, (-|-)/\ker(\pi), \pi(\check{\Pi}_{\text{nor}})) \quad \text{and} \quad (\pi^*(\bar{\mathfrak{a}}^*) = \iota^*(V), (-|-)/\ker(\iota^*), \iota^*(\Pi_{\text{nor}}))$$

are root bases. In the non-affine case, the respective Coxeter systems resulting from Proposition A.9 are isomorphic to the Coxeter system (W, S) introduced in Definition A.5.

Proof. One computes

$$(\check{n}_i|\check{n}_i) = \left(\frac{1}{\sqrt{2\varepsilon_i}}\check{\alpha}_i \middle| \frac{1}{\sqrt{2\varepsilon_i}}\check{\alpha}_i\right) = \frac{1}{2\varepsilon_i}(\check{\alpha}_i|\check{\alpha}_i) = \frac{1}{2\varepsilon_i}b_{ii}\varepsilon_i\varepsilon_i = \frac{1}{2\varepsilon_i}a_{ii}\varepsilon_i = 1$$

and

$$(\check{n}_i|\check{n}_j) = \left(\frac{1}{\sqrt{2\varepsilon_i}}\check{\alpha}_i \middle| \frac{1}{\sqrt{2\varepsilon_j}}\check{\alpha}_j\right) = \frac{1}{2\sqrt{\varepsilon_i\varepsilon_j}}(\check{\alpha}_i|\check{\alpha}_j) = \frac{1}{2}\frac{\sqrt{\varepsilon_j}}{\sqrt{\varepsilon_i}}a_{ij} = -\frac{1}{2}\sqrt{\frac{a_{ji}}{a_{ij}}}|a_{ij}| = -\frac{1}{2}\sqrt{a_{ij}a_{ji}}.$$

Moreover,

$$(n_i|n_i) = \left(\frac{\sqrt{\varepsilon_i}}{\sqrt{2}}\alpha_i \middle| \frac{\sqrt{\varepsilon_i}}{\sqrt{2}}\alpha_i\right) = \frac{\varepsilon_i}{2}(\alpha_i|\alpha_i) = 1$$

and

$$(n_i|n_j) = \left(\frac{\sqrt{\varepsilon_i}}{\sqrt{2}}\alpha_i \middle| \frac{\sqrt{\varepsilon_j}}{\sqrt{2}}\alpha_j\right) = \frac{\sqrt{\varepsilon_i\varepsilon_j}}{2}(\alpha_i|\alpha_j) = \frac{1}{2}\frac{\sqrt{\varepsilon_j}}{\sqrt{\varepsilon_i}}a_{ij} = -\frac{1}{2}\sqrt{\frac{a_{ji}}{a_{ij}}}|a_{ij}| = -\frac{1}{2}\sqrt{a_{ij}a_{ji}}.$$

It follows that $(\check{n}_i|\check{n}_j), (n_i|n_i) \in \{-\cos(\pi/m) \mid m \in \mathbb{N}\} \cup]-\infty, -1]$. Altogether, $(\mathfrak{a}, (-|-), \check{\Pi}_{\text{nor}})$ and $(V, (-|-), \Pi_{\text{nor}})$ satisfy axioms (i) and (ii) of the definition of a root basis. Linear independence of $\check{\Pi}_{\text{nor}}$ and Π_{nor} furthermore imply axiom (iii). Thus $(\mathfrak{a}, (-|-), \check{\Pi}_{\text{nor}})$ and $(V, (-|-), \Pi_{\text{nor}})$

are root bases, and in view of Proposition A.9 it follows from the explicit formulas (A.10) and (A.11) that the corresponding Coxeter systems are isomorphic to (W, S) .

Equality (A.10) moreover implies that the radical of the invariant bilinear form on \mathfrak{a} equals $\ker(\pi) = \mathfrak{c} \cap \mathfrak{a}$ (see also [Kac90, Lemma 2.1]). Equality (A.11) implies that the radical of the invariant bilinear form on V equals $\ker(\iota^*)$, as for any $\lambda \in \ker(\iota^*)$ one has $r_{\alpha_i}(\lambda) = \lambda - \lambda(\check{\alpha}_i)\alpha_i = \lambda$ and $\dim_{\mathbb{R}} \ker(\iota^*) = n - l$, where l is the rank of \mathbf{A} . Thus if \mathbf{A} is non-affine, then [Kra09, Proposition 6.1.3] applies, and the images of $(\mathfrak{a}, (-|-), \check{\Pi}_{\text{nor}})$ and $(V, (-|-), \Pi_{\text{nor}})$ on $\bar{\mathfrak{a}}$ and $\iota^*(V)$ are root bases as well, for the same Coxeter system. \square

In the sequel denote the bilinear form $(-|-)/\ker(\pi)$ on $\bar{\mathfrak{a}}$ simply by $(-|-)$. Note that this form is always non-degenerate, since the radical of the invariant bilinear form on \mathfrak{a} equals $\ker(\pi) = \mathfrak{c} \cap \mathfrak{a} < \{h \in \mathfrak{h}(\mathbf{A}) \mid \forall i = 1, \dots, n : \alpha_i(h) = 0\}$. Also write $\sigma_i := \bar{\rho}_{KM}(\check{r}_{\alpha_i})$ for the Coxeter generators of $\bar{\rho}_{KM}(W)$.

Corollary A.11. *If \mathbf{A} is symmetrizable and non-affine, then $(\bar{\mathfrak{a}}, (-|-), \pi(\check{\Pi}_{\text{nor}}))$ is a root basis for the Coxeter system $(\bar{\rho}_{KM}(W), \{\sigma_1, \dots, \sigma_n\}) \cong (W, S)$ and the reduced Kac-Moody representation $\bar{\rho}_{KM} : W \rightarrow \text{GL}(\bar{\mathfrak{a}})$ is faithful.* \square

Note that the statement of the corollary does not hold in the affine case. Here the image $\bar{\rho}_{KM}(W)$ is just the canonical finite quotient of W given by the underlying spherical Coxeter diagram, and thus the reduced Kac-Moody representation is *not* faithful.

Corollary A.12. *If \mathbf{A} is symmetrizable and non-affine, then the linear map*

$$\begin{aligned} \bar{\varphi} : (\pi(\mathfrak{a}) = \bar{\mathfrak{a}}, (-|-)/\ker(\pi), \pi(\check{\Pi}_{\text{nor}})) &\rightarrow (\iota^*(\bar{\mathfrak{a}}^*) = \iota^*(V), (-|-)/\ker(\iota^*), \iota^*(\Pi_{\text{nor}})) \\ \pi(\check{n}_j) &\mapsto \iota^*(n_j) \end{aligned}$$

is an isometry. Furthermore,

$$(\pi(\check{n}_i)|\pi(\check{n}_j)) = (\iota^*(n_i)|\iota^*(n_j)) = n_j(\check{n}_i).$$

Proof. Note that the family $(\pi(\check{n}_j))_{1 \leq j \leq n}$ is not necessarily linearly independent and so, a priori, it is not even clear that there exists a linear map at all such that $\pi(\check{n}_j) \mapsto \iota^*(n_j)$. However, there certainly exists a linear map

$$\begin{aligned} \varphi : (\mathfrak{a}, (-|-), \check{\Pi}_{\text{nor}}) &\rightarrow (V, (-|-), \Pi_{\text{nor}}) \\ \check{n}_j &\mapsto n_j. \end{aligned}$$

By the computation in the proof of Proposition A.10 one has

$$(\varphi(\check{n}_i)|\varphi(\check{n}_j)) = (n_i|n_j) = \frac{1}{2} \frac{\sqrt{\varepsilon_j}}{\sqrt{\varepsilon_i}} a_{ij} = \frac{1}{2} \frac{\sqrt{\varepsilon_j}}{\sqrt{\varepsilon_i}} \alpha_j(\check{\alpha}_i) = n_j(\check{n}_i).$$

By that proof, moreover, $\ker(\pi)$ equals the radical of the bilinear form on \mathfrak{a} and $\ker(\iota^*)$ equals the radical of the bilinear form on V , so that factoring out the respective radicals induces the desired isometry between $\pi(\mathfrak{a})$ and $\iota^*(V)$. \square

AF The reduced Tits cone and the Coxeter complex

Assume that the generalized Coxeter matrix \mathbf{A} is symmetrizable and non-affine. In analogy with [Kac90, §3.12] define the following:

Definition A.13. The (closed) *fundamental geometric chamber* of $\bar{\mathfrak{a}}$ is the subset

$$C := \{h \in \bar{\mathfrak{a}} \mid \alpha_i(h) \geq 0 \text{ for } 1 \leq i \leq n\}.$$

The union

$$\mathcal{C} := \bigcup_{w \in W} w(C) \subset \bar{\mathfrak{a}}$$

is called the *reduced Tits cone*. Connected components of $\mathcal{C} \cap \bar{\mathfrak{a}}^{\text{reg}}$, i.e. geometric chambers contained in the Tits cone, are called (open) *Tits chambers*.

Remark A.14. (i) Under the isometric isomorphism from Corollary A.12 the reduced Tits cone $\mathcal{C} \subset \bar{\mathfrak{a}}$ corresponds to a *dual reduced Tits cone* $\mathcal{C}^* \subset \bar{\mathfrak{a}}^*$, which in turn embeds into \mathfrak{a}^* under the map π^* from (A.7). The resulting cone in \mathfrak{a}^* is precisely the Tits cone from [Kac90, §3.12]. Thus, up to duality between roots and coroots, while preserving forms and factoring out the radicals of these forms, our definition of the Tits cone actually coincides with Kac's definition. Therefore any of Kac's results concerning the Tits cone that are invariant under form-equivariant linear maps in fact also hold true in our setting.

(ii) Since every simple root reflection turns precisely one positive root negative, one has the characterization

$$\mathcal{C} = \{X \in \bar{\mathfrak{a}} \mid \alpha(X) \geq 0 \text{ for almost all } \alpha \in \Phi^+\};$$

cf. [Kac90, Proposition 3.12(c)].

It is useful to consider the reduced Tits cone as a simplicial complex with simplices given by intersections of closures of Tits chambers. For the following corollary, recall from [AB08, Definition 3.1] the definition of the Coxeter complex $\Sigma = \Sigma(W, S)$ of the pair (W, S) . By [AB08, Theorem 3.5] this is a simplicial complex, and by [Dav08, Section 4.4] its dual graph is given by the Cayley graph $\text{Cay}(W, S)$. Fix a fundamental chamber C_o of Σ and obtain an identification of the faces of C_o with fundamental reflections. Denote by H_i the wall of the fundamental chamber C_o corresponding to the fundamental reflection \check{r}_{α_i} .

Corollary A.15. *If \mathbf{A} is symmetrizable and non-affine, then there is a unique simplicial isomorphism $\bar{\varphi} : \Sigma \rightarrow \mathcal{C}$ between the Coxeter complex Σ and the reduced Tits cone \mathcal{C} with the following properties:*

- (i) $\bar{\varphi}$ is $\bar{\rho}_{KM}$ -equivariant.
- (ii) $\bar{\varphi}$ maps the fundamental chamber C_o of Σ to the fundamental chamber C of \mathcal{C} .
- (iii) $\bar{\varphi}(H_i) = \bar{H}_{\alpha_i}$.

Proof. Recall that for every root α , the kernel of the reflection $\sigma_\alpha := \bar{\rho}_{KM}(\check{r}_\alpha)$ is given by the hyperplane $\bar{H}_\alpha \subset \bar{\mathfrak{a}}^{\text{sing}}$. Since $(-|-)$ is non-degenerate and W -invariant, the reflection σ_α is in fact the unique $(-|-)$ -orthogonal reflection at the hyperplane \bar{H}_α . The map $\sigma_\alpha \mapsto \bar{H}_\alpha$ thus provides a bijection between reflections in $\bar{\rho}_{KM}(W) \cong W$ and hyperplanes in $\bar{\mathfrak{a}}^{\text{sing}}$. In other words, the reduced Tits cone is a simplicial complex on which W acts by simplicial automorphisms in such a way that faces of chambers are fixpoint sets of reflections in W . Then the corollary follows from [Dav08, Theorem 3.3.4]. \square

Again, this fails in the affine case, where the reduced Tits cone is instead isomorphic to the Coxeter complex of the spherical quotient.

AG Automorphisms of the Coxeter complex acting on the Tits cone

Keep the assumption \mathbf{A} is a symmetrizable and non-affine generalized Coxeter matrix with associated Coxeter system (W, S) . Denote by $\Sigma = \Sigma(W, S)$ the Coxeter complex of (W, S) and by $\text{Aut}(\Sigma)$ the group of simplicial automorphisms of the Coxeter complex. Equivalently, one can think of $\text{Aut}(\Sigma)$ as the automorphisms of the Cayley graph $\text{Cay}(W, S)$ (not necessarily preserving the coloring) or as the automorphism group of the Coxeter diagram underlying (W, S) .

Denote by $\text{Aut}(W, S) < \text{Aut}(W)$ the subgroup of automorphisms of W which preserve S as a set. This subgroup acts faithfully by automorphisms on the Cayley graph of (W, S) and thus $\text{Aut}(W, S) < \text{Aut}(\Sigma)$; in fact, this group equals the automorphism group of the Coxeter diagram. Also, W acts by automorphisms on Σ and thus can be considered as a subgroup of $\text{Aut}(\Sigma)$.

Lemma A.16 ([AB08, 3.34, 3.35]). *The automorphism group $\text{Aut}(\Sigma)$ splits as a semidirect product $\text{Aut}(\Sigma) = W \rtimes \text{Aut}(W, S)$. Moreover, $\text{Aut}(W, S)$ is isomorphic to the group of automorphisms of the Coxeter diagram of (W, S) .* \square

Lemma A.16 enables one to extend the reduced Kac–Moody representation $\bar{\rho}_{KM} : W \rightarrow \text{GL}(\bar{\mathfrak{a}})$ to $\text{Aut}(W, S)$. Indeed, every diagram automorphism $\alpha \in \text{Aut}(W, S)$ corresponds to a permutation of the walls of the fundamental chamber which preserves angles. Any such permutation can be realized by a unique linear map $\bar{\alpha}$ of the ambient vector space $\bar{\mathfrak{a}}$. One thus obtains a monomorphism

$$\bar{\rho} : \text{Aut}(\Sigma) = W \rtimes \text{Aut}(W, S) \rightarrow \text{GL}(\bar{\mathfrak{a}})$$

which maps each diagram automorphism α to $\bar{\alpha}$ and restricts to $\bar{\rho}_{KM}$ on S . Refer to $\bar{\rho}$ as the *canonical linear realization* of $\text{Aut}(\Sigma)$ over \mathfrak{a} . By construction, this representation takes values in the group

$$\text{GL}(\bar{\mathfrak{a}}, \bar{\mathfrak{a}}^{\text{sing}}) := \{f \in \text{GL}(\bar{\mathfrak{a}}) \mid f(\bar{\mathfrak{a}}^{\text{sing}}) = \bar{\mathfrak{a}}^{\text{sing}}\}$$

of those linear automorphisms of $\bar{\mathfrak{a}}$ which preserve the hyperplane arrangement $\bar{\mathfrak{a}}^{\text{sing}}$. This hyperplane arrangement is also invariant under $-\text{Id}_{\bar{\mathfrak{a}}}$, which may or may not be contained in the image of $\bar{\rho}$. One can thus extend the canonical linear realization to a homomorphism

$$\bar{\rho} : \text{Aut}(\Sigma) \times \mathbb{Z}/2\mathbb{Z} \rightarrow \text{GL}(\bar{\mathfrak{a}}),$$

by letting the generator of $\mathbb{Z}/2\mathbb{Z}$ act by $-\text{Id}_{\bar{\mathfrak{a}}}$. One then has the following rigidity result, which was pointed out to us by Bernhard Mühlherr.

Proposition A.17 (Mühlherr, personal communication). *Let \mathbf{A} be a non-affine irreducible generalized Cartan matrix of size $n \times n$ with $n \geq 2$, let (W, S) be the associated Coxeter system and let $\Sigma = \Sigma(W, S)$ be the associated Coxeter complex. Then the canonical linear realization defines a surjective homomorphism*

$$\bar{\rho} : \text{Aut}(\Sigma) \times \mathbb{Z}/2\mathbb{Z} \rightarrow \text{GL}(\bar{\mathfrak{a}}, \bar{\mathfrak{a}}^{\text{sing}}).$$

If $-\text{Id}_{\bar{\mathfrak{a}}} \notin \bar{\rho}(\text{Aut}(\Sigma))$, then this map is an isomorphism.

Proof. Let $\varphi \in \text{GL}(\bar{\mathfrak{a}}, \bar{\mathfrak{a}}^{\text{sing}})$. First establish that φ normalizes $\bar{W} := \bar{\rho}_{KM}(W)$ and that conjugation by φ preserves reflections in \bar{W} . To this end, as before denote by $\sigma_i := \bar{\rho}_{KM}(r_{\alpha_i})$ the orthogonal reflection at the hyperplane $\bar{H}_i := \bar{H}_{\alpha_i}$. Recall that the hyperplanes $\bar{H}_1, \dots, \bar{H}_n$ bound the fundamental chamber $C \in \mathcal{C}$.

Since the pair $(\bar{W}, \{\sigma_1, \dots, \sigma_n\})$ is a Coxeter system, its conjugate $(\bar{W}^\varphi, \{\sigma_1^\varphi, \dots, \sigma_n^\varphi\})$ by φ is also a Coxeter system. Each σ_i^φ is a reflection, because it has a 1-eigenspace of codimension 1 and is of order 2. It follows that all reflections of the Coxeter system $(\bar{W}^\varphi, \{\sigma_1^\varphi, \dots, \sigma_n^\varphi\})$ act by reflections (not necessarily orthogonal) on $\bar{\mathfrak{a}}$. These reflections preserve $\bar{\mathfrak{a}}^{\text{sing}}$, since φ does. Moreover, every hyperplane in $\bar{\mathfrak{a}}^{\text{sing}}$ is the set of fixed points of a unique reflection in \bar{W}^φ , since $\varphi(\bar{\mathfrak{a}}^{\text{sing}}) = \bar{\mathfrak{a}}^{\text{sing}}$. In particular for every $i = 1, \dots, n$ there is a unique reflection $\tilde{\sigma}_i$ in \bar{W}^φ with fixed-point set \bar{H}_i .

Note that, by definition, $\tilde{\sigma}_i$ exchanges i -adjacent Tits chambers. In particular, both σ_i and $\tilde{\sigma}_i$ map the fundamental chamber C to its unique i -adjacent chamber. It follows that for $i = 1, \dots, n$ the linear map $\tilde{\sigma}_i \sigma_i^{-1}$ preserves the hyperplane \bar{H}_i pointwise and the fundamental chamber C setwise. Since \mathbf{A} is irreducible with $n \geq 2$, the product $\tilde{\sigma}_i \sigma_i^{-1}$ therefore fixes a basis of $\bar{\mathfrak{a}}$ and hence $\tilde{\sigma}_i = \sigma_i$ for all $i = 1, \dots, n$. In particular, $\bar{W} = \langle \tilde{\sigma}_1, \dots, \tilde{\sigma}_n \rangle$ is a subgroup of \bar{W}^φ .

The reflections $\{\tilde{\sigma}_1, \dots, \tilde{\sigma}_n\}$ actually generate \overline{W}^φ . Indeed, since \overline{W}^φ is generated by reflections at certain hyperplanes \overline{H}_α , it will suffice to show that $\overline{W} = \langle \tilde{\sigma}_1, \dots, \tilde{\sigma}_n \rangle$ contains reflections at all such hyperplanes. Since \overline{W} acts sharply transitively on the Coxeter complex of (W, S) , it acts sharply transitively on chambers in the reduced Tits cone. In particular, it contains reflections at all hyperplanes in $\overline{\mathfrak{a}}^{\text{sing}}$ which intersect the Tits cone. Since in fact every wall in $\overline{\mathfrak{a}}^{\text{sing}}$ intersects the Tits cone, one deduces that $\overline{W} = \overline{W}^\varphi$.

That is, φ normalizes \overline{W} . Moreover, φ maps fundamental reflections, and thus arbitrary reflections, to reflections. Denote by $\overline{\varphi}$ the automorphisms of \overline{W} induced by conjugation with φ . Since $\overline{\varphi}$ preserves reflections, it maps root bases to root bases. By [Bou02, Sec. VI.1.5], the group W acts transitively on such root bases. One may thus assume that $\overline{\varphi}$ stabilizes S and thus induces an automorphism α of Σ . Then $\overline{\rho}(\alpha)$ agrees with φ up to $\pm \text{Id}_{\overline{\mathfrak{a}}}$. Indeed, by definition φ and $\overline{\rho}(\alpha)$ are both linear maps preserving the hyperplane arrangement $\mathfrak{a}^{\text{sing}}$ and (since every hyperplane intersects the Tits cone) they map each hyperplane in $\mathfrak{a}^{\text{sing}}$ to the same hyperplane. This is only possible if they are either equal or differ by a global minus sign. \square

Remark A.18. One has the following trichotomy:

- (i) If \mathbf{A} is spherical, then $-C \subset C = \overline{\mathfrak{a}}$ and thus $-\text{Id}_{\overline{\mathfrak{a}}} \in \overline{\rho}(\text{Aut}(\Sigma))$. In this case, $\overline{\rho}$ yields an isomorphism $\text{Aut}(\Sigma) \cong \text{GL}(\overline{\mathfrak{a}}, \overline{\mathfrak{a}}^{\text{sing}})$.
- (ii) If \mathbf{A} is non-spherical and non-affine, then $-C$ is not contained in C . Indeed, by definition, all $\alpha \in \Phi^+$ take non-positive values on $-C$. Since Φ^+ contains infinitely many elements, one has $-C \not\subset C$ in view of Remark A.14(ii). In this case there exist two embeddings of the Coxeter complex Σ into $\overline{\mathfrak{a}}$, which are given by the Tits cone C and its negative $-C$. Note that the intersection of these two cones is given by

$$C \cap -C = \{X \in \overline{\mathfrak{a}} \mid \alpha(X) = 0 \text{ for almost all } \alpha \in \Phi^+\} = \{0\}.$$

The action of $\overline{\rho}(\text{Aut}(\Sigma))$ on $\overline{\mathfrak{a}}$ preserves C and $-C$, whereas $-\text{Id}_{\overline{\mathfrak{a}}}$ exchanges the two cones. In particular, $\overline{\rho}$ induces an isomorphism

$$\text{Aut}(\Sigma) \times \mathbb{Z}/2\mathbb{Z} \cong \text{GL}(\overline{\mathfrak{a}}, \overline{\mathfrak{a}}^{\text{sing}})$$

in this case.

- (iii) If \mathbf{A} is affine, then the action of W on $\overline{\mathfrak{a}}$ is not faithful, and the W -module $\overline{\mathfrak{a}}$ is given by the Kac–Moody representation of the underlying spherical Coxeter system (W_o, S_o) . In this case one, thus, has $\text{GL}(\overline{\mathfrak{a}}, \overline{\mathfrak{a}}^{\text{sing}}) \cong \text{Aut}(\Sigma(W_o, S_o))$ by (i).

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